Part I: Multiple Choice (5 points each)

There is no partial credit. You may not use a calculator.

1. Find the area of the region bounded by the curves

$$x = 0, \quad x = 1 + y^2, \quad y = 1, \quad y = 3.$$

(A) 11

(B) $\dfrac{40}{3}$

(C) 12

(D) $\dfrac{32}{3}$ $\Leftarrow$ correct

(E) 10

[Draw a sketch!]

$$\begin{aligned} A &= \int_1^3 (1 + y^2)\, dy \\ &= \left[y + \frac{y^3}{3} \right]_1^3 \\ &= (3 + 9) - \left(1 + \tfrac{1}{3}\right) \\ &= 11 - \tfrac{1}{3} \\ &= \frac{32}{3}\,. \end{aligned}$$

2. Find the average value of the function $f(x) = e^{-3x}$ on the interval $[0, 2]$.

(A) $\frac{1}{6}(1 - e^{-6})$ $\Leftarrow$ correct

(B) $\frac{3}{2}(1 + e^{6})$

(C) 0

(D) $\frac{1}{3}(1 - e^{-6})$

(E) $\frac{1}{2}(e^{-6} - 1)$

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{2 - 0} \int_0^2 e^{-3x}\, dx \\ &= \frac{1}{2}\, \frac{e^{-3x}}{-3} \bigg|_0^2 \\ &= -\frac{1}{6}\left[e^{-6} - e^{0}\right] \\ &= \frac{1}{6} - \frac{1}{6} e^{-6}. \end{aligned}$$

3. Calculate $\displaystyle\int_0^{\pi/4} \sin^2 x\, dx$.

(A) $\dfrac{\pi}{8} - \dfrac{1}{2}$

(B) $\dfrac{\pi}{4} - \dfrac{1}{2}$

(C) $\dfrac{\pi}{8} - \dfrac{1}{4}$ ⇐ correct

(D) $\dfrac{\pi}{8} + \dfrac{1}{2}$

(E) $\dfrac{\pi}{4} + \dfrac{1}{4}$

$$\begin{aligned}\int_0^{\pi/4} \sin^2 x\, dx &= \int_0^{\pi/4} \frac{1-\cos 2x}{2}\, dx \\ &= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/4} \\ &= \frac{1}{2}\left(\frac{\pi}{4} - \frac{\sin(\pi/2)}{2}\right) \\ &= \frac{\pi}{8} - \frac{1}{4}.\end{aligned}$$

4. An object is moved along the x-axis by a force of magnitude $F(x) = \dfrac{1}{1+x^2}$. How much work is done as the object moves from $x = 0$ to $x = 1$?

(A) π

(B) $\dfrac{\pi}{16}$

(C) $\ln 2$

(D) $\dfrac{\pi}{4}$ ⇐ correct

(E) $\ln 8 - \ln 2$

$$\begin{aligned}W &= \int_0^1 \frac{1}{1+x^2}\, dx \\ &= \arctan x \Big|_0^1 \\ &= \arctan 1 \\ &= \frac{\pi}{4}.\end{aligned}$$

5. The area bounded by the curves $x^2 = y$ and $x + y = 2$ is

(A) 5

(B) $\frac{3}{2}$

(C) $\frac{9}{2}$ $\Leftarrow$ correct

(D) π

(E) $\frac{\pi}{2}$

[Draw a sketch!]

Find the intersections: $x^2 = y = 2 - x$.

$$0 = x^2 + x - 2 = (x-1)(x+2).$$

$$x = -2 \text{ or } 1.$$

$$\begin{aligned} A &= \int_{-2}^{1} \bigl((2-x) - x^2\bigr)\,dx \\ &= \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^{1} \\ &= \bigl(2 - \tfrac{1}{2} - \tfrac{1}{3}\bigr) - \bigl(-4 - 2 + \tfrac{8}{3}\bigr) \\ &= 8 - \tfrac{1}{2} - \tfrac{9}{3} = \tfrac{9}{2}\,. \end{aligned}$$

6. A trigonometric substitution converts the integral $\displaystyle\int \frac{x}{(3-2x-x^2)^{1/2}}dx$ to

(A) $\int (3\cos\theta + 2)\,d\theta$

(B) $\int (2\sin\theta - 1)\,d\theta$ $\Leftarrow$ correct

(C) $\int (2\sin^2\theta - \cos\theta)\,d\theta$

(D) $\int (2\tan\theta - 1)\,d\theta$

(E) $\int 2\tan^{-1}\theta\,d\theta$

First complete the square:

$$\begin{aligned} -(x^2 + 2x - 3) &= -(x^2 + 2x + 1 - 4) \\ &= -\bigl((x+1)^2 - 4\bigr). \end{aligned}$$

In $\displaystyle\int \frac{x}{\sqrt{4-(x+1)^2}}\,dx$ let $x + 1 = 2\sin\theta$.

$$dx = 2\cos\theta\,d\theta, \quad x = 2\sin\theta - 1.$$

$$\int \frac{2\sin\theta - 1}{\sqrt{4 - 4\sin^2\theta}}\,2\cos\theta\,d\theta = \int (2\sin\theta - 1)\,d\theta.$$

7. Suppose that $f(0) = 3$ and $f(2) = 4$ and $\int_0^2 x^2 f(x)\,dx = 5$. What is $\int_0^2 x^3 f'(x)\,dx$? (*Hint:* Use integration by parts. Assume that $f(x)$ is a differentiable function and that $f'(x)$ is continuous.)

(A) 60

(B) 47

(C) 33

(D) 27

(E) 17 $\Leftarrow$ correct

Let $u = x^3$, $dv = f'(x)\,dx$.

$$du = 3x^2\,dx, \quad v = f(x).$$

The integral becomes

$$\begin{aligned} &\left[x^3 f(x)\right]_0^2 - 3\int_0^2 x^2 f(x)\,dx \\ =&8f(2) - 0f(0) - 3\int_0^2 x^2 f(x)\,dx \\ =&8 \times 4 - 3 \times 5 = 17. \end{aligned}$$

8. The region bounded by the curves $x = 0$, $x = 1+y$, $y = 0$, $y = 2$ is rotated about the y-axis. Find the volume of the resulting solid.

(A) $\frac{26\pi}{3}$ $\Leftarrow$ correct

(B) $\frac{80\pi}{3}$

(C) $\frac{22\pi}{3}$

(D) $\frac{32\pi}{3}$

(E) $\frac{62\pi}{3}$

[Draw a sketch!]

The disk at y has radius $r = x = 1 + y$.

$$\begin{aligned} V &= \int_0^2 \pi r^2\,dy \\ &= \pi \int_0^2 (1 + 2y + y^2)\,dy \\ &= \pi \left[y + y^2 + \frac{y^3}{3}\right]_0^2 \\ &= \pi\left(2 + 4 + \frac{8}{3}\right) = \frac{26\pi}{3}. \end{aligned}$$

9. Calculate $\displaystyle\int_1^e \frac{\ln x}{x^2}\,dx$.

(A) $e^2 - 1$

(B) 0

(C) 1

(D) $1 - \dfrac{2}{e}$ ⇐ correct

(E) $2 - e^2$

Let $u = \ln x$, $dv = \dfrac{1}{x^2}\,dx$.

$$du = \frac{1}{x}\,dx, \quad v = -\frac{1}{x}.$$

The integral becomes

$$-\frac{1}{x}\ln x + \int \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\ln x - \frac{1}{x}\right]_1^e$$
$$= \left(-\frac{1}{e} - \frac{1}{e}\right) - (-0 - 1)$$
$$= 1 - \frac{2}{e}.$$

10. The base of a solid is the triangle with vertices $(0,0)$, $(1,1)$, and $(1,-1)$. The cross sections perpendicular to the x-axis are squares. Find the volume.

(A) $\dfrac{1}{3}$

(B) $\dfrac{2}{3}$

(C) $\dfrac{4}{3}$ ⇐ correct

(D) $\dfrac{16}{3}$

(E) $\dfrac{32}{3}$

[Draw a sketch of the base.]

The top and bottom boundaries of the base are

$$y = x \quad \text{and} \quad y = -x.$$

So the square at x has side length $s = 2y = 2x$.

$$V = \int_0^1 s^2\,dx$$
$$= \int_0^1 4x^2\,dx$$
$$= 4\left.\frac{x^3}{3}\right|_0^1$$
$$= \frac{4}{3}.$$

Part II: Write Out (10 points each)

Show all your work. Appropriate partial credit will be given. You may not use a calculator.

11. Evaluate $\displaystyle\int x^2 \sin(4x)\, dx$.

Method 1: Let $u = x^2$, $dv = \sin 4x\, dx$; $du = 2x\, dx$, $v = -\dfrac{\cos 4x}{4}$.
This converts the integral to

$$-\frac{x^2}{4}\cos 4x + \frac{1}{2}\int x\cos 4x\, dx.$$

Integrate by parts again: $u = x$, $dv = \cos 4x\, dx$; $du = dx$, $v = \dfrac{\sin 4x}{4}$.
The integral becomes

$$\begin{aligned} &-\frac{x^2}{4}\cos 4x + \frac{1}{2}\left[\frac{x}{4}\sin 4x - \frac{x}{4}\int \sin 4x\, dx\right] \\ &= -\frac{x^2}{4}\cos 4x + \frac{1}{2}\left[\frac{x}{4}\sin 4x - \frac{1}{4}\left(-\frac{\cos 4x}{4}\right)\right] + C \\ &= -\frac{x^2}{4}\cos 4x + \frac{x}{8}\sin 4x + \frac{1}{32}\cos 4x + C. \end{aligned}$$

Method 2: First make the substitution $y = 4x$ to simplify the arithmetic.

$$dx = \tfrac{1}{4}\, dy, \quad x^2 = \tfrac{1}{16}\, y^2.$$

$$\int x^2 \sin 4x\, dx = \frac{1}{64}\int y^2 \sin y\, dy.$$

Now let $u = y^2$, $dv = \sin y\, dy$; $du = 2y\, dy$, $v = -\cos y$.
This converts the integral $\int y^2 \sin y\, dy$ to $-y^2\cos y + 2\int y\cos y\, dy$.
Integrate by parts again: $u = y$, $dv = \cos y\, dy$; $du = dy$, $v = \sin y$.
We get

$$-y^2\cos y + 2\left[y\sin y - \int \sin y\, dy\right] = -y^2\cos y + 2y\sin y + 2\cos y + C.$$

So the original integral is

$$\begin{aligned} &\frac{1}{64}\left[-(4x)^2\cos 4x + 2(4x)\sin 4x + 2\cos 4x + C\right] \\ &= -\frac{x^2}{4}\cos 4x + \frac{x}{8}\sin 4x + \frac{1}{32}\cos 4x + C. \end{aligned}$$

12. Evaluate $\int \frac{\sin^3 x}{\cos^4 x}\,dx$.

Let $u \cos x$, so that $du = -\sin x\,dx$ and $\sin^2 x = 1 - u^2$. The integral is then

$$\begin{aligned} -\int \frac{1-u^2}{u^4}\,du &= -\int \frac{1}{u^4}\,du + \int \frac{1}{u^2}\,du \\ &= \left[\frac{1}{3}\frac{1}{u^3} - \frac{1}{u}\right] + C \\ &= \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C. \end{aligned}$$

13. Set up (but DO NOT EVALUATE) the integrals to compute the volumes of the indicated solids of revolution. CLEARLY INDICATE IN EACH CASE WHETHER YOU ARE WRITING A CYLINDER-SHELL FORMULA OR A DISKS/WASHERS FORMULA.

(a) Revolve the region bounded by $y = \sin x$, $y = 0$, $x = 0$, $x = \pi$ about the line $x = 0$ (the y-axis).

[Draw a sketch!]

Cylinder method (best): $V = \int 2\pi r\,h\,dx = 2\pi \int_0^{\pi} x \sin x\,dx$

Washer method (ugly): $V = \int_0^1 \pi[(\pi - \arcsin y)^2 - (\arcsin y)^2]\,dy$

(b) Revolve the region bounded by $y = \sin x$, $y = 0$, $x = 0$, $x = \pi$ about the line $y = 2$.

[Draw a sketch!]

Washer method (best):

$$V = \int \left(\pi r_{\text{outer}}^2 - \pi r_{\text{inner}}^2\right) dx = \pi \int_0^{\pi} [4 - (2 - \sin x)^2]\,dx = \pi \int_0^{\pi} (4 \sin x - \sin^2 x)\,dx$$

Cylinder method (ugly): $V = \int_0^1 2\pi(2 - y)[(\pi - \arcsin y) - \arcsin y]\,dy$

14. Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$.

Let $x = 3\sec\theta$, so that $dx = 3\sec\theta\tan\theta$ and $x^2 - 9 = 9\tan^2\theta$. The integral becomes

$$\begin{aligned}\int \frac{3\sec\theta\tan\theta}{3\tan\theta}\,d\theta &= \int \sec\theta\,d\theta \\ &= \ln|\sec\theta + \tan\theta| + C \\ &= \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C.\end{aligned}$$

[Sketch a right triangle with hypotenuse x, adjacent side 3, and therefore opposite side $\sqrt{x^2-9}$.]

15. Do **ONE** of the following [(A) or (B)]. CIRCLE THE LETTER of the one you want graded!

(A) A 10-kilogram object at ground level is attached by a cable with a mass density of $\frac{1}{4}$ kg/m to a winch at the top of a 40-meter high building. How much work (in joules) is required to crank this load up to the roof? (The acceleration of gravity in MKS units is $g = 9.8$.)

The work done in lifting the object itself is just the force times the distance:

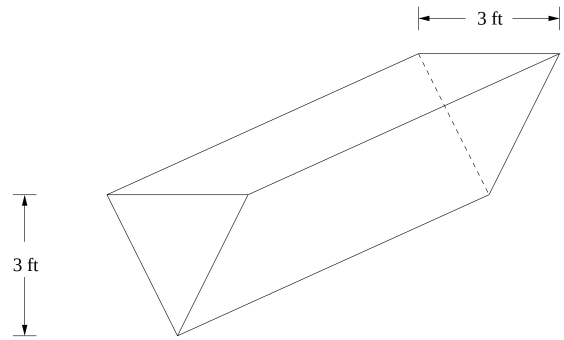
$$W_1 = 10g \times 40 = 400g.$$

The work done in lifting the piece of cable of length dy that starts at distance y below the roof is

$$W_2 = \int_0^{40} \frac{1}{4}\,g\,y\,dy = \frac{g}{4}\,\frac{y^2}{2}\bigg|_0^{40} = \frac{g}{8}\,40 \times 40 = 200g.$$

So the total work is $W = 600g = 5880\text{J}$.

(B) A tank (trough) 8 feet long has cross sections that are isosceles triangles (with base side on top) whose base and altitude are both 3 feet. If the tank is initially full of water, how much work is required to pump all the water out over the top? (Water weighs 62.5 pounds per cubic foot.)



$W = \int D\,dF$, where $D = 3-y$ is the depth of the layer of water at y and $dF = \rho g\,dV = (\rho g) \times 8w\,dy$ is the gravitational force (weight) on that layer. Here w is the width of the trough at height y, and $\rho g = 62.5$ is the weight density. By similar triangles,

$$\frac{w}{y} = \frac{3}{3},$$

it is clear that $w = y$. Thus

$$\begin{aligned} W &= \int_0^3 8y(3-y)(\rho g)\,dy = 8(\rho g)\int_0^3 3y - y^2)\,dy \\ &= 8(\rho g)\left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 = 8(\rho g)\left(\frac{27}{2} - 9\right) = \frac{72}{2}(\rho g) \\ &= 36 \times 62.5 = 2250\text{ft} - \text{lb}. \end{aligned}$$