Part I: Multiple Choice (5 points each)

There is no partial credit. You may not use a calculator.

1. A cube, with edges of length 2 meters, sits on the flat bottom of a pool of water that is 5 meters deep. Find the force due to water pressure on any one vertical side of the cube. (Let ρ be the mass density of water and g be the acceleration of gravity; $\rho = 1000$ kg/m^3 and $g = 9.8$ m/s^2, but you don't need to use that.)

(A) $20\rho g$

(B) $50\rho g$

(C) $4\rho g$

(D) $12\rho g$

(E) $16\rho g$ $\Leftarrow$ correct

$$\begin{aligned} F &= \int_{\text{bottom}}^{\text{top}} \rho g\,(\text{depth})(\text{width})\,d(\text{height}) \\ &= \int_0^2 \rho g(5-y)2\,dy \\ &= 2\rho g\left[5y - \tfrac{y^2}{2}\right]_0^2 \\ &= 2\rho g(10-2) = 16\rho g. \end{aligned}$$

2. Find the force due to water pressure on the top surface of the cube in the previous problem.

(A) $20\rho g$

(B) $50\rho g$

(C) $4\rho g$

(D) $12\rho g$ $\Leftarrow$ correct

(E) $16\rho g$

By definition of pressure,

$$\begin{aligned} F &= \text{force} \cdot \text{area} \\ &= \rho g \cdot 3 \cdot 4 \\ &= 12\rho g. \end{aligned}$$

3. The improper integral $\displaystyle\int_0^4 \frac{dx}{\sqrt{x}}$

(A) converges to the value 4 $\Leftarrow$ correct

(B) diverges, by comparison with the integral $\displaystyle\int_0^4 \frac{dx}{x^2}$

$$\lim_{T\to 0^+} 2\sqrt{x}\Big|_T^4 = 4.$$

(C) converges to the value 3

(D) converges to the value $\frac{1}{2}$

(E) converges, by comparison with the integral $\displaystyle\int_0^4 \frac{dx}{x^2}$

4. Set up the integral for the arc length of the curve $y = \dfrac{x^3}{6}$ between $x = 1$ and $x = 2$.

(A) $\displaystyle\frac{\pi}{36}\int_1^2 x^6\,dx$

(B) $\displaystyle\int_1^2 \sqrt{1+\frac{x^4}{4}}\,dx$ $\Leftarrow$ correct

(C) $\displaystyle\int_1^2 \sqrt{1+\frac{x^2}{2}}\,dx$

(D) $\displaystyle\frac{\pi}{3}\int_1^2 x^4\,dx$

(E) $\displaystyle\int_1^2 \sqrt{1+\frac{x^6}{36}}\,dx$

$$\frac{dy}{dx} = \frac{3x^2}{6} = \frac{x^2}{2}.$$

$$L = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 \sqrt{1+\frac{x^4}{4}}\,dx.$$

5. The improper integral $\displaystyle\int_e^{\infty} \frac{dx}{x \ln x}$ [**Hint:** Let $u = \ln x$.]

(A) converges to the value -1

(B) converges to the value 0

(C) converges, by comparison with the integral $\displaystyle\int_e^{\infty} \frac{dx}{x}$

(D) converges to the value $-e$

(E) diverges to $+\infty$ $\Leftarrow$ correct

$$du = \frac{1}{x}\,dx.$$

$$\begin{aligned} I &= \int_1^{\infty} \frac{du}{u} \\ &= \lim_{T\to\infty} \ln u \Big|_1^T \\ &= \infty. \end{aligned}$$

6. Set up the integral for the arc length of the parametrized curve segment

$$x(t) = \sin 2t, \quad y(t) = 3\cos 2t, \qquad 0 \le t \le \frac{\pi}{4}.$$

(A) $\displaystyle\int_0^{\pi/4} \left(1 + \sqrt{4\cos^2 2t + 36\sin^2 2t}\right) dt$

(B) $\displaystyle\int_0^{\pi/4} \sqrt{1 + 2\cos 2t + 9\sin 2t}\, dt$

(C) $\displaystyle\int_0^{\pi/4} \sqrt{4\cos^2 2t + 36\sin^2 2t}\, dt$ $\Leftarrow$ correct

(D) $\displaystyle\int_0^{\pi/4} \sqrt{1 + \sin^2 2t + 9\cos^2 2t}\, dt$

(E) $\displaystyle\int_0^{\pi/4} \sqrt{4\sin^2 2t + 6\cos^2 2t}\, dt$

$$\frac{dx}{dt} = 2\cos 2t,$$

$$\frac{dy}{dt} = -6\sin 2t.$$

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \text{response (C)}. \end{aligned}$$

7. To find the partial-fraction decomposition of $\dfrac{3x^4+9x^3+9x^2+x+8}{(x^2+4x+5)(x-1)^2(x+2)}$ you would start from the form

(A) $\dfrac{A}{x+2}+\dfrac{B}{(x-1)^2}+\dfrac{Cx+D}{x^2+4x+5}+\dfrac{Ex+F}{(x^2+4x+5)^2}$

(B) $\dfrac{A}{x+2}+\dfrac{B}{x-1}+\dfrac{Cx+D}{x^2+4x+5}$

(C) $\dfrac{A}{x+2}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x-1)}+\dfrac{Dx+E}{x^2+4x+5}$ $\Leftarrow$ correct

(D) $\dfrac{A}{x+2}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x^2+4x+5}$

(E) $3x^2+\dfrac{A}{x+2}+\dfrac{B}{(x-1)^2}+\dfrac{Cx+D}{x^2+4x+5}$

8. An integrating factor for the differential equation $\dfrac{dy}{dx}+\dfrac{3y}{x}=\cos x$ is

(A) $e^{\sin x}$

(B) x^3 $\Leftarrow$ correct

(C) $3x$

(D) $3\ln x$

(E) $e^{3/x}$

$$P(x)=\frac{3}{x}.$$

$$\begin{aligned} I &= e^{\int P\,dx} \\ &= e^{3\ln x} \\ &= x^3. \end{aligned}$$

9. The improper integral $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^{3/2}}\,dx$

(A) converges, by comparison with the integral $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\,dx$ $\Leftarrow$ correct

(B) diverges, by comparison with the integral $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\,dx$

(C) diverges, by comparison with the integral $\displaystyle\int_1^\infty \sin^2 x\,dx$

(D) converges to the value 0

(E) converges to the value $-\pi$

Because $\sin^2 x \le 1$, we can compare the integral with $\int_1^\infty x^{-3/2}\,dx$ provided that the latter converges.

$$\int_1^\infty \frac{1}{x^{3/2}}\,dx = \left.\frac{2x^{-1/2}}{-1}\right|_1^\infty = 0 + 2 < \infty.$$

10. Suppose that $y(t)$ is the solution to the initial value problem $\dfrac{dy}{dt} = y^2, \quad y(0) = \dfrac{1}{2}$. What is $y(5)$?

(A) $-\dfrac{1}{3}$ $\Leftarrow$ correct

(B) $-\dfrac{1}{5}$

(C) $\dfrac{1}{7}$

(D) $\dfrac{1}{2}e^5$

(E) $\dfrac{1}{2}e^{5/2}$

Separable equation:

$$\int \frac{dy}{y^2} = \int dt \Rightarrow -\frac{1}{y} = t + C.$$

$$-\frac{1}{1/2} = 0 + C \Rightarrow C = -2.$$

$$\frac{1}{y} = 2 - t \Rightarrow y = \frac{1}{2-t}.$$

$$y(5) = \frac{1}{2-5} = -\frac{1}{3}.$$

Part II: Write Out (10 points each)

Show all your work. Appropriate partial credit will be given. You may not use a calculator.

11. The curve $y = \sqrt{x}$, between $x = 0$ and $x = 4$, is rotated about the x-axis. Find the surface area of the resulting surface.

$$A = \int_0^4 2\pi\sqrt{x}\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\,dx = 2\pi \int_0^4 \sqrt{x + \frac{1}{4}}\,dx$$
$$= 2\pi\ \tfrac{2}{3}\left(x + \tfrac{1}{4}\right)^{3/2}\Big|_0^4 = \tfrac{4\pi}{3}\left[\left(4 + \tfrac{1}{4}\right)^{3/2} - \left(\tfrac{1}{4}\right)^{3/2}\right]$$
$$= \frac{4\pi}{3}\,\frac{17^{3/4} - 1}{8} = \frac{\pi}{6}\left(17^{3/2} - 1\right).$$

12. A tank has volume 1000 liters and initially contains pure fresh water. Water containing 10 grams of salt per liter runs into the tank at a rate of 20 L/min. The salt water mixes instantly and completely with the fresh water, and the mixture drains out the bottom at a rate of 20 L/min. Find a formula for $S(t)$, the number of grams of salt in the tank after t minutes.

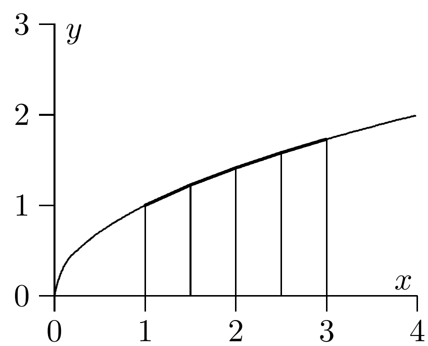
$$\frac{dS}{dt} = \text{salt flow in} - \text{salt flow out}$$
$$= 10 \cdot 20 - \frac{S}{1000} \cdot 20 = 200 - \frac{1}{50}S.$$
$$\frac{dS}{dt} + \frac{1}{50}S = 200.$$
$$P = \frac{1}{50} \ \Rightarrow\ I = e^{\int \frac{1}{50}\,dt} = e^{t/50}.$$
$$e^{t/50} \cdot 200 = e^{t/50}\frac{dS}{dt} + \frac{1}{50}e^{t/50}S = \frac{d}{dt}\left(e^{t/50}S\right).$$
$$e^{t/50}S = \int 200e^{t/50}\,dt = 10\,000e^{t/50} + C.$$
$$S(t) = 10\,000 + Ce^{-t/50}.$$
$$0 = S(0) = 10\,000 + C \ \Rightarrow\ C = -10\,000.$$
$$S(t) = 10\,000\left(1 - e^{-t/50}\right).$$

13. In this problem we shall use the trapezoid rule with $n = 4$ to approximate $\int_1^3 \sqrt{x}\,dx$. The three parts of the problem can be done in any order. Note: At no time should you be using the antiderivative of the function $\sqrt{x}$.

(a) Write out the trapezoid-rule approximation T_4 to $\displaystyle\int_1^3 \sqrt{x}\,dx$.
DO NOT SIMPLIFY; the purpose of this question is to show that you understand the formula, not that you can do arithmetic.

$$\begin{aligned} T_4 &= \left(\tfrac{1}{2}f(1) + f(1.5) + f(2) + f(2.5) + \tfrac{1}{2}f(3)\right)\frac{3-1}{4} \\ &= \tfrac{1}{2}\left(\tfrac{1}{2}\sqrt{1} + \sqrt{1.5} + \sqrt{2} + \sqrt{2.5} + \tfrac{1}{2}\sqrt{3}\right) \end{aligned}$$

(b) Sketch the graph of $f(x) = \sqrt{x}$, $1 \le x \le 3$, along with the approximating trapezoids.



(c) Using the error-bound formula

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{where } |f''(x)| \le K \text{ for } a \le x \le b,$$

find an upper bound for the error in using T_4 to approximate $\displaystyle\int_1^3 \sqrt{x}\,dx$.

$$f(x) = x^{1/2} \Rightarrow f''(x) = -\frac{1}{4}x^{-3/2}.$$

The maximum value of $|f''|$ occurs at the left end, $x = 1$. Therefore, we can take $K = \frac{1}{4}$. Thus

$$|E_T| \le \frac{\frac{1}{4}\cdot 2^3}{12 \cdot 4^2} = \frac{1}{96}.$$

14. Evaluate $\displaystyle\int \frac{3x^2 - x + 8}{x^3 + 4x}\, dx$.

Perform a partial-fraction decomposition:

$$\frac{3x^2 - x + 8}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}\,.$$

$$3x^2 - x + 8 = A(x^2 + 4) + (Bx + C)x = Ax^2 + 4A + Bx^2 + Cx.$$

Setting $x = 0$ yields $4A = 8$, or $A = 2$. There are no other "special" values of x in this problem, so one should either choose two other values at random, or (more easily) just equate the coefficients of x and of x^2 on the two sides of the equation:

$$3 = A + B = 2 + B, \qquad -1 = C.$$

Thus $B = 1$ and $C = -1$;

$$\frac{3x^2 - x + 8}{x^3 + 4x} = \frac{2}{x} + \frac{x - 1}{x^2 + 4}\,.$$

So the integral is

$$\int \frac{2}{x}\, dx + \int \frac{x - 1}{x^2 + 4}\, dx = 2\ln|x| + \frac{1}{2}\ln(x^2 + 4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

15. A plate of uniform density occupies the region bounded by $y = 1 + x^3$, $y = 0$, $x = 0$, and $x = 2$. Find $\overline{x}$, the x component of the centroid (center of mass) of the plate.

Let ρ be the density. First find the mass of the plate:

$$M = \int_0^2 \rho(1 + x^3)\, dx = \rho\left[x + \frac{x^4}{4}\right]_0^2 = \rho(2 + 4 - 0 - 0) = 6\rho.$$

Then the first moment:

$$M_1 = \int_0^2 \rho x(1 + x^3)\, dx = \rho\left[\frac{x^2}{2} + \frac{x^5}{5}\right]_0^2 = \rho\left(2 + \frac{32}{5} - 0 - 0\right) = \frac{42}{5}\,\rho.$$

Then

$$\bar{x} = \frac{M_1}{M} = \frac{42\rho}{5 \cdot 6\rho} = \frac{7}{5}\,.$$

Note: The constant density cancels out of the calculation. Therefore, one can leave it out from the beginning (in problems where it is a constant); in that case, M is the area of the plate (not the mass).