Part I: Multiple Choice (4 points each)

There is no partial credit. You may not use a calculator.

1. Suppose that the nth partial sum of the series $\sum_{n=1}^{\infty} a_n$ is given by $s_n = 1 - \dfrac{\ln n}{n}$. Then the series $\sum_{n=1}^{\infty} a_n$

   (A) converges to 0.

   (B) converges to 1. $\Leftarrow$ correct

   (C) converges to -1.

   (D) diverges to $+\infty$.

   (E) diverges to $-\infty$.

The sum of the series is the limit of the sequence of partial sums.

$$\begin{aligned}\lim_{n\to\infty} s_n &= 1 - \lim_{n\to\infty} \frac{\ln n}{n}\\ &= 1 - \lim_{n\to\infty} \frac{1/n}{1}\\ &= 1.\end{aligned}$$

2. If $\vec{a} = \langle 1, 1, 1\rangle$ and $\vec{b} = \langle 1, -2, 3\rangle$, then $3\vec{a} + \vec{b}$ is

   (A) $\langle 0, 3, -2\rangle$

   (B) $\langle 2, -1, 4\rangle$

   (C) $\langle 2, 1, 3\rangle$

   (D) $\langle 4, 2, 0\rangle$

   (E) $\langle 4, 1, 6\rangle$ $\Leftarrow$ correct

$$\begin{aligned}&\langle 3, 3, 3\rangle + \langle 1, -2, 3\rangle\\ &\qquad = \langle 3+1, 3-2, 3+3\rangle.\end{aligned}$$

3. $\lim_{n\to\infty} \frac{\cos(n^{97})}{\sqrt{n}}$ equals

(A) 0 $\Leftarrow$ correct

(B) ± 1

(C) 97

(D) $\frac{97}{2}$

(E) It does not exist.

$$\lim_{n\to\infty} \left| \frac{\cos(n^{97})}{\sqrt{n}} \right| \le \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Thus the elements of the sequence approach 0.

4. Compute the sum of the series $\sum_{n=0}^{\infty} \frac{2^n}{5^n}$, if it converges.

(A) $\frac{5}{3}$ $\Leftarrow$ correct

(B) $\frac{2}{5}$

(C) $\frac{5}{2}$

(D) $\frac{3}{5}$

(E) It diverges.

This is the geometric series with $r = \frac{2}{5}$ (and initial factor $a = 1$). Its sum is

$$\frac{1}{1 - \frac{2}{5}} = \frac{5}{5-2} = \frac{5}{3}.$$

5. The region of $\mathbf{R}^3$ represented by the equation $xyz = 0$ consists of

(A) the x-axis, the y-axis, and the z-axis.

(B) the x-axis and the yz-plane.

(C) the xy-plane, the yz-plane, and the xz-plane. $\Leftarrow$ correct

(D) a line.

(E) none of these.

The product is zero if and only if one of the factors is zero:

$$x = 0 \quad \text{or} \quad y = 0 \quad \text{or} \quad z = 0.$$

Each of these loci is one of the coordinate planes ($x = 0$ precisely on the y–z plane, etc.).

6. The Maclaurin series (Taylor series around 0) of $f(x) = x^2 e^{-3x}$ is

(A) $\displaystyle\sum_{n=0}^{\infty} \frac{-x^{n+2}}{3^n n!}$

(B) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{3^n n!}$

(C) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^{n+2}}{n!}$ $\Leftarrow$ correct

(D) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{(3n)!}$

(E) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{n+2}}{(3n)!}$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

$$e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{n!}.$$

$$x^2 e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3)^n x^{n+2}}{n!}.$$

7. The series $\displaystyle\sum_{n=1}^{\infty} \frac{n + e^{-2n}}{n^2 - e^{-n}}$ is

(A) convergent, by the ratio test.

(B) divergent, by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.
$\Leftarrow$ correct

(C) convergent, by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

(D) divergent, by comparison with $\displaystyle\sum_{n=1}^{\infty} e^n$.

(E) convergent, by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n}$.

If we drop the exponential terms, the numerator becomes smaller and the denominator becomes larger. So the nth term is larger than

$$\frac{n}{n^2} = \frac{1}{n}.$$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so the original series diverges by the comparison test. (The limit comparison test could also be used.)

8. The sequence $\left\{\dfrac{n^2-1}{n}\right\}_{n=1}^{\infty}$ is

   (A) increasing and bounded above.

   (B) decreasing and bounded above.

   (C) increasing and bounded below. ⇐ correct

   (D) decreasing and bounded below.

   (E) none of these

$$\frac{n^2-1}{n} = n - \frac{1}{n}\,.$$

The first few numbers are $1-1=0$, $2-\frac{1}{2}=\frac{3}{2}$, …, and it is clear that at each further step n adds more than $1/n$ takes away, so the sequence is increasing. The sequence is bounded below by its first element, 0. (And it is not bounded above, since $\lim\left(1-\frac{1}{n}\right)=+\infty$.)

9. The series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ is

   (A) absolutely convergent.

   (B) divergent, because the exponent $\frac{1}{2}$ is less than 1.

   (C) convergent, by the ratio test.

   (D) convergent, by the alternating series test. ⇐ correct

   (E) divergent, by the alternating series test.

The signs suggest trying the alternating series test. We must check its other two conditions: $\frac{1}{\sqrt{n}}$ is decreasing, and its limit is 0. So the test applies; the series converges.

Note that $\sum\frac{1}{\sqrt{n}}$ is a divergent p-series, so our series is not absolutely convergent.

10. What is the Maclaurin series of $f(x) = x^2$?

(A) $x^2 + x^3 + x^4 + \cdots$

(B) x^2 $\Leftarrow$ correct

(C) $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$

(D) $\sum_{n=0}^{\infty} (x-1)^2$

(E) It does not exist.

Nobody ever said a Taylor series had to have more than one term. Every polynomial is already a Maclaurin series. (To check the answer, calculate all the derivatives of $f(x) = x^2$ and evaluate them at 0. Only the second derivative survives.)

11. Use the Maclaurin series (Taylor series around 0) for e^{-x^3} to compute

$$\lim_{x\to 0} \frac{e^{-x^3} - 1 + x^3}{x^6}.$$

(A) $-\frac{3}{6!}$

(B) $\frac{3}{6!}$

(C) $\frac{1}{6!}$

(D) $-\frac{1}{2}$

(E) $\frac{1}{2}$ $\Leftarrow$ correct

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots.$$

$$e^{-x^3} = 1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \cdots.$$

$$\lim_{x\to 0} \frac{\left(1 - x^3 + \frac{x^6}{2} - \frac{x^9}{6} + \cdots\right) - 1 + x^3}{x^6}$$

$$= \frac{\frac{x^6}{2} - \frac{x^9}{6} + \cdots}{x^6}$$

$$= \frac{1}{2}.$$

12. The Taylor series of $f(x) = \dfrac{1}{x}$ around (i.e., centered at) the point $x = -3$ is

(A) $-\sum_{n=0}^{\infty} \dfrac{(x+3)^n}{3^{n+1}}$ ⇐ correct

(B) $\sum_{n=0}^{\infty} \dfrac{x^n}{3^{n+1}}$

(C) $\sum_{n=0}^{\infty} \dfrac{(-1)^n (x-3)^n}{3^n}$

(D) $\sum_{n=0}^{\infty} (x+3)^n$

(E) $\sum_{n=0}^{\infty} (-3)^n x^n$

The easy way:

$$\frac{1}{x} = \frac{1}{(x+3)-3} = -\frac{1}{3}\,\frac{1}{1-\frac{(x+3)}{3}} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x+3}{3}\right)^n \quad \text{(geometric series).}$$

Main step of the hard way:

$$f^{(n)}(x) = (-1)\frac{n!}{x^{n+1}} \Rightarrow f^{(n)}(-3) = -\frac{n!}{3^{n+1}}\,.$$

13. Use the Maclaurin series for $\cos(t^2)$ to find the Maclaurin series for $\int_0^x \cos(t^2)\,dt$.

(A) $-\dfrac{4x^3}{2!} + \dfrac{8x^7}{4!} - \cdots$

(B) $\dfrac{x^2}{2\cdot 0!} - \dfrac{x^6}{6\cdot 4!} + \dfrac{x^{10}}{10\cdot 8!} - \cdots$

(C) $x - \dfrac{x^3}{3!} + \dfrac{x^7}{7!} - \cdots$

(D) $x - \dfrac{x^5}{5\cdot 2!} + \dfrac{x^9}{9\cdot 4!} - \cdots$ ⇐ correct

(E) $x - \dfrac{x^3}{3\cdot 2!} + \dfrac{x^5}{5\cdot 4!} - \cdots$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots.$$

$$\cos t^2 = 1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \cdots.$$

$$\int_0^x \cos t^2\,dt = x - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \cdots.$$

Part II: Write Out (10 points each)

Show all your work. Appropriate partial credit will be given. You may not use a calculator.

14. Find the radius of convergence and interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{5^n(n+1)}$.
(Full credit requires determining what happens at the endpoints of the interval.)

Apply the ratio test:

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x-3|^{n+1}}{5^{n+1}(n+2)} \frac{5^n(n+1)}{|x-3|^n}$$
$$= \lim_{n\to\infty} \frac{|x-3|^{n+1}}{|x-3|^n} \frac{5^n}{5^{n+1}} \frac{n+1}{n+2} = \frac{|x-3|}{5} \lim_{n\to\infty} \frac{n+1}{n+2} = \frac{|x-3|}{5}.$$

The series converges if $L < 1$; that is, $|x-3| < 5$. Thus the radius of convergence is $R = 5$ and the open interval of convergence (centered on $a = 3$) is $-2 < x < 8$.

It remains to check what happens at the endpoints.

$x = -2$: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ CONVERGES (alternating series test).

$x = 8$: $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n+1}$ DIVERGES (harmonic series).

In summary, the series converges on the interval $-2 \le x < 8$, or $[-2, \infty)$.

15. For what values of t is the angle between the vectors $\langle t, 1, 1\rangle$ and $\langle t, t, -1\rangle$ equal to 90° ?

Perpendicularity is equivalent to vanishing of the dot product:

$$\langle t, 1, 1\rangle \cdot \langle t, t, -1\rangle = 0.$$

$$t^2 + t - 1 = 0.$$

$$t = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

16. Obtain an upper bound for the error in approximating each of these series by its 5th partial sum, s_5 . (You are not asked to prove that the series converge, but thinking about why that is true will help you to answer the question. Also, you may take it to be well known that xe^{-x^2} is a decreasing function for $x \geq 1$.)

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n^2}$

This series satisfies the alternating series test, so the error is less than the absolute value of the first term omitted:

$$|r_5| < 6e^{-6^2} = 6e^{-36}.$$

(Here r_5 means $s - s_5 = \displaystyle\sum_{n=6}^{\infty} a_n$.)

(b) $\displaystyle\sum_{n=1}^{\infty} n e^{-n^2}$

This series converges by the integral test, so the error is bounded by replacing the omitted tail of the series by the corresponding integral:

$$r_5 = |r_5| < \int_5^\infty x e^{-x^2}\, dx = \left. \frac{e^{-x^2}}{-2} \right|_5^\infty = \frac{1}{2} e^{-25}.$$

17. Determine whether or not the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!\, 3^n}{(2n)!}$ converges, giving clear justification for your answer.

This looks like a good candidate for the ratio test.

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!\, 3^{n+1}}{(2n+2)!} \frac{(2n)!}{n!\, 3^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{(2n)!}{(2n+2)!} \frac{3^{n+1}}{3^n} = \lim_{n\to\infty} \frac{(n+1)3}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{3}{2(2n+1)} = 0.$$

Since all we needed was $L < 1$, the series certainly converges.

18. Let $f(x) = \sqrt{x}$.

(a) Approximate $f(x)$ by a Taylor polynomial of degree 2 centered at the point $a = 9$.

$$f(x) = x^{1/2}, \quad f'(x) = \frac{1}{2}\,x^{-1/2}, \quad f''(x) = -\,\frac{1}{4}\,x^{-3/2}.$$

$$f(9) = 3, \quad f'(9) = \frac{1}{6}\,, \quad f''(9) = -\,\frac{1}{4 \cdot 27}\,.$$

$$\begin{aligned} f(x) &\approx f(9) + f'(9)(x-9) + \frac{1}{2}f''(9)(x-9)^2 \\ &= 3 + \frac{1}{6}(x-9) - \frac{1}{8 \cdot 27}(x-9)^2. \end{aligned}$$

(b) Find an upper bound on the error of this approximation for all x in the interval $7 \le x \le 11$. *Hint:* The Taylor remainder inequality is

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1},$$

where M is an upper bound on $|f^{(n+1)}(c)|$ for all c in the interval concerned.

We need to calculate one more derivative:

$$f^{(3)}(x) = \frac{3}{8}\,x^{-5/2}.$$

This is a decreasing positive function, so its maximum absolute value occurs at the beginning of the interval, $x = 7$.

$$M = f^{(3)}(7) = \frac{3}{8 \cdot 7^{5/2}}\,.$$

The largest value of $|x-9|^3$ also occurs at the endpoints, where $|x-9| = 2$. Thus

$$|R_2(x)| \le \frac{3 \cdot 2^3}{3!\,2^3 7^{5/2}} = \frac{1}{2 \cdot 7^{5/2}}\,.$$