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MATH 152 | Final Exam | Spring 2001

Sections 201,202 | Solutions | P. Yasskin

Multiple Choice: (4 points each)

1. Compute $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n}$

a. e^{-2} correctchoice
b. 1
c. $e^{1/2}$
d. e^{2}
e. ∞

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{2n} = \exp\lim_{n\to\infty}\ln\left(1-\frac{1}{n}\right)^{2n} = \exp\lim_{n\to\infty}\frac{\ln\left(1-\frac{1}{n}\right)}{\frac{1}{2n}} = \exp\lim_{n\to\infty}\frac{\frac{\frac{1}{n^2}}{1-\frac{1}{n}}}{\frac{-1}{2n^2}}$$

$$= \exp\lim_{n\to\infty}\frac{-2}{1-\frac{1}{n}} = e^{-2}$$

2. Compute $\int_0^{\pi/4}\sin^2\theta\cos^2\theta\,d\theta$

a. $\frac{\pi}{32}$ correctchoice
b. $\frac{\pi}{16}$
c. $\frac{\pi}{8}$
d. $\frac{\pi}{4}$
e. $\frac{\pi}{2}$

$$\int_0^{\pi/4}\sin^2\theta\cos^2\theta\,d\theta = \frac{1}{4}\int_0^{\pi/4}\sin^2(2\theta)\,d\theta = \frac{1}{4}\int_0^{\pi/4}\frac{1-\cos(4\theta)}{2}\,d\theta = \frac{1}{8}\left[\theta-\frac{\sin(4\theta)}{4}\right]_0^{\pi/4} = \frac{\pi}{32}$$

3. The region below $y=\frac{1}{x}$ above the x-axis between $x=1$ and $x=\infty$ is rotated about the x-axis. Find the volume of the solid of revolution.

a. $\frac{\pi}{4}$
b. $\frac{\pi}{2}$
c. π correctchoice
d. 2π
e. 4π

$$\int_1^{\infty}\pi\left(\frac{1}{x}\right)^2dx = \left[\frac{-\pi}{x}\right]_1^{\infty} = \pi$$

4. Compute $\int_0^1 x^2 e^{-x}\,dx$

a. $-5e^{-1}$
b. $2-5e^{-1}$ correctchoice
c. $-e^{-1}$
d. $2-e^{-1}$
e. $e^{-1}-2$

$u = x^2 \qquad dv = e^{-x}dx$
$du = 2x\,dx \qquad v = -e^{-x}$

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} + 2\int x e^{-x}\,dx$$

$u = x \qquad dv = e^{-x}dx$
$du = dx \qquad v = -e^{-x}$

$$\int x^2 e^{-x}\,dx = -x^2 e^{-x} + 2\left(-xe^{-x} + \int e^{-x}\,dx\right)$$

$$\int_0^1 x^2 e^{-x}\,dx = \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^1 = \left[-e^{-1} - 2e^{-1} - 2e^{-1}\right] - [-2] = 2 - 5e^{-x}$$

5. Find the average value of the function $f(x) = \dfrac{1}{(x^2+9)^{3/2}}$ on the interval $[0,3]$.

a. $\dfrac{\sqrt{2}}{6}$
b. $\dfrac{\sqrt{2}}{27}$
c. $\dfrac{\sqrt{2}}{9}$
d. $\dfrac{1}{9\sqrt{2}}$
e. $\dfrac{1}{27\sqrt{2}}$ correctchoice

$f_{\text{ave}} = \dfrac{1}{3}\displaystyle\int_0^3 \frac{dx}{(x^2+9)^{3/2}}$ Let $x = 3\tan\theta$. Then $dx = 3\sec^2\theta\,d\theta$

$$f_{\text{ave}} = \frac{1}{3}\int_0^{\pi/4} \frac{3\sec^2\theta\,d\theta}{(9\tan^2\theta+9)^{3/2}} = \frac{1}{27}\int_0^{\pi/4}\frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^{3/2}} = \frac{1}{27}\int_0^{\pi/4}\cos\theta\,d\theta = \left[\frac{1}{27}\sin\theta\right]_0^{\pi/4} = \frac{1}{27\sqrt{2}}$$

6. Find the angle between the vector $\vec{u} = (2,1,-2)$ and the normal to the plane through $P = (3,-4,12)$ containing the vectors $\vec{v} = (1,0,0)$ and $\vec{w} = (0,-3,4)$.

a. $\arccos\left(\dfrac{-22}{39}\right)$
b. $\arccos\left(\dfrac{15}{2}\right)$
c. $\arccos\left(\dfrac{3}{2}\right)$
d. $\arccos\left(\dfrac{2}{15}\right)$ correctchoice
e. $\arccos\left(\dfrac{2}{3}\right)$

$\vec{N} = \vec{v}\times\vec{w} = (1,0,0)\times(0,-3,4) = (0,-4,-3) \qquad \left|\vec{N}\right| = \sqrt{16+9} = 5$

$\vec{u}\cdot\vec{N} = (2,1,-2)\cdot(0,-4,-3) = 2 \qquad |\vec{u}| = \sqrt{4+1+4} = 3 \qquad \cos\theta = \dfrac{\vec{u}\cdot\vec{N}}{|\vec{u}|\left|\vec{N}\right|} = \dfrac{2}{15}$

7. Use the 4th degree Maclaurin polynomial for e^{-x^2} to estimate $\int_0^1 e^{-x^2}\,dx$.

- **a.** $1 - \frac{1}{3} + \frac{1}{5}$
- **b.** $1 + \frac{1}{3} + \frac{1}{5}$
- **c.** $1 - \frac{1}{6} + \frac{1}{120}$
- **d.** $1 + \frac{1}{3} + \frac{1}{10}$
- **e.** $1 - \frac{1}{3} + \frac{1}{10}$ correctchoice

$e^t \approx 1 + t + \frac{t^2}{2} \qquad e^{-x^2} \approx 1 - x^2 + \frac{x^4}{2}$

$\int_0^1 e^{-x^2}\,dx \approx \int_0^1 \left(1 - x^2 + \frac{x^4}{2}\right) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10}\right]_0^1 = 1 - \frac{1}{3} + \frac{1}{10}$

8. The series $\sum_{n=1}^{\infty} \frac{n}{n^{3/2}+1}$ is

- **a.** convergent by the Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$
- **b.** conv. by the Limit Comp. Test with $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ but not by the Comp. Test
- **c.** divergent by the Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$
- **d.** div. by the Limit Comp. Test with $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ but not by the Comp. Test

  correctchoice
- **e.** none of these

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is divergent because it is a p-series with $p = \frac{1}{2} < 1$

$\frac{n}{n^{3/2}+1} < \frac{1}{n^{1/2}}$ So the Comp. Test cannot show convergence or divergence.

$\lim_{n\to\infty} \frac{\frac{n}{n^{3/2}+1}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{n^{3/2}}{n^{3/2}+1} = 1$ So $\sum_{n=1}^{\infty} \frac{n}{n^{3/2}+1}$ is divergent by the Limit Comp. Test

9. The area below $y = x^2$, above the x-axis, between $x = 1$ and $x = 2$ is rotated about the y-axis. Find the volume of the solid of revolution.

- **a.** 4π
- **b.** $\frac{15\pi}{4}$
- **c.** $\frac{15\pi}{2}$ correctchoice
- **d.** 8π
- **e.** $\frac{31\pi}{5}$

x-integral cylinders $r = x \quad h = y = x^2$

$V = \int 2\pi r h\,dx = \int_1^2 2\pi x x^2 dx = 2\pi\left[\frac{x^4}{4}\right]_1^2 = 8\pi - \frac{\pi}{2} = \frac{15\pi}{2}$

10. Compute $\displaystyle\sum_{n=2}^{\infty}\left(\frac{n+1}{n-1}-\frac{n+2}{n}\right)$

a. 0
b. 1
c. 2 correctchoice
d. 3
e. divergent

$$S_k=\sum_{n=2}^{k}\left(\frac{n+1}{n-1}-\frac{n+2}{n}\right)=\left(\frac{3}{1}-\frac{4}{2}\right)+\left(\frac{4}{2}-\frac{5}{3}\right)+\cdots+\left(\frac{k}{k-2}-\frac{k+1}{k-1}\right)+\left(\frac{k+1}{k-1}-\frac{k+2}{k}\right)$$
$$=\frac{3}{1}-\frac{k+2}{k}$$
$$S=\lim_{k\to\infty}S_k=\lim_{k\to\infty}\left(3-\frac{k+2}{k}\right)=2$$

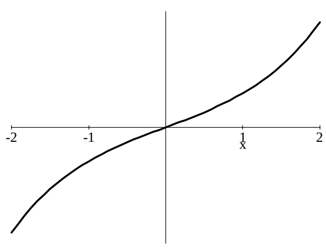
11. The Maclaurin series for $\sinh x$ is

$$\sinh x=\sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots$$

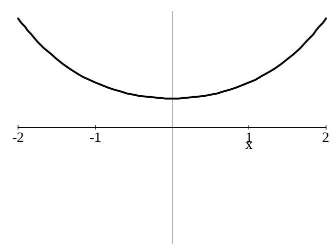
If you use the 5^{th}-degree Maclaurin polynomial to approximate $\sinh x$ on the interval $\left[\frac{1}{2},2\right]$, bound the error in the approximation using the Taylor Remainder Inequality

$$|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1}\qquad\text{where}\quad M\ge|f^{(n+1)}(c)|\quad\text{for all } c \text{ between } x \text{ and } a.$$

HINT: $\sinh x$:



$\cosh x$:



a. $\frac{4}{15}\cosh 2$
b. $\frac{4}{45}\sinh 2$ correctchoice
c. $\frac{4}{45}\cosh\frac{1}{2}$
d. $\frac{4}{15}\sinh\frac{1}{2}$
e. $\frac{4}{45}\cosh 2$

Here, $n=5$, $a=0$, and x is in the interval $\left[\frac{1}{2},2\right]$. The maximum value of $|x-a|$ is 2.

$f(x)=\sinh x \quad f'(x)=\cosh x \quad f''(x)=\sinh x \quad f^{(3)}(x)=\cosh x$

$f^{(4)}(x)=\sinh x \quad f^{(5)}(x)=\cosh x \quad f^{(6)}(x)=\sinh x$

$M\ge\sinh c$ for c between 0 and 2 and $\sinh x$ is increasing. So take $M=\sinh 2$.

$$|R_n(x)|\le\frac{M}{(n+1)!}|x-a|^{n+1}\le\frac{\sinh 2}{6!}2^6=\frac{4}{45}\sinh 2$$

12. Find the point (a,b,c) where the line $x = 2-t \quad y = 3+2t \quad z = 4+t$ intersects the plane $2x - y + 3z = 14$. Then $a+b+c =$

a. 1
b. 3
c. 5
d. 7 correctchoice
e. 9

$2(2-t) - (3+2t) + 3(4+t) = 13 - t = 14 \quad \Rightarrow \quad t = -1$
$x = 2-t = 3 \quad y = 3+2t = 1 \quad z = 4+t = 3 \quad \Rightarrow \quad (a,b,c) = (3,1,3) \quad \Rightarrow \quad a+b+c = 7$

Work Out (13 points each)

Show all your work. Partial credit will be given. You may not use a calculator.

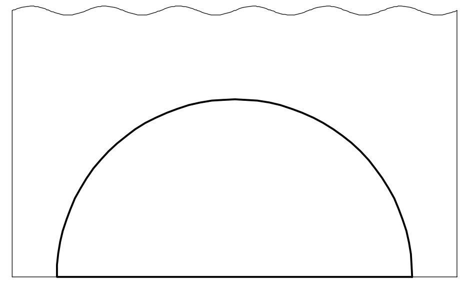
13. Find the solution of the differential equation $x^3 \dfrac{dy}{dx} - 2y = 4$ satisfying the initial condition $y(1) = 3$.

$\dfrac{dy}{dx} - \dfrac{2}{x^3}y = \dfrac{4}{x^3} \qquad P = \dfrac{-2}{x^3} \qquad I = e^{\int P\,dx} = e^{1/x^2}$
$e^{1/x^2}\dfrac{dy}{dx} - \dfrac{2}{x^3}e^{1/x^2}y = \dfrac{4}{x^3}e^{1/x^2} \qquad \dfrac{d}{dx}\left(e^{1/x^2}y\right) = \dfrac{4}{x^3}e^{1/x^2}$
$e^{1/x^2}y = \int \dfrac{4}{x^3}e^{1/x^2}\,dx = -2e^{1/x^2} + C \qquad x = 1, y = 3 \qquad e3 = -2e + C \qquad C = 5e$
$e^{1/x^2}y = -2e^{1/x^2} + 5e \qquad y = -2 + 5e^{1-1/x^2}$

14. The curve $y = x^2$ for $0 \le x \le \sqrt{2}$ is rotated about the y-axis. Find the surface area of the resulting surface.

$A = \int 2\pi r\,ds \qquad r = x \qquad ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx = \sqrt{1+(2x)^2}\,dx = \sqrt{1+4x^2}\,dx$
$A = \int_0^{\sqrt{2}} 2\pi x\sqrt{1+4x^2}\,dx = \dfrac{\pi}{6}(1+4x^2)^{3/2}\Big|_0^{\sqrt{2}} = \dfrac{\pi}{6}(9)^{3/2} - \dfrac{\pi}{6}(1)^{3/2} = \dfrac{13\pi}{3}$

15. A plate in the shape of a semicircle is placed at the bottom if a tank with the straight edge down. The radius of the circle is 4 cm and the water in the tank is 6 cm deep.
What is the force on the plate?
(The density of water is $\rho = 1000 \frac{\text{kg}}{\text{m}^3}$ and the acceleration of gravity is $g = 9.8 \frac{\text{m}}{\text{sec}^2}$, but you may leave your answer in terms of ρg.)



Measure y up from the bottom of the tank. The slice at height y has width $w = 2x = 2\sqrt{16 - y^2}$. The water above this slice has depth $h = 6 - y$. So the force is

$$F = \int \rho g h \, dA = \rho g \int_0^4 (6 - y) 2\sqrt{16 - y^2} \, dy = 12\rho g \int_0^4 \sqrt{16 - y^2} \, dy - 2\rho g \int_0^4 y\sqrt{16 - y^2} \, dy$$

$\int_0^4 \sqrt{16 - y^2} \, dy =$ Area of a quarter circle of radius $4 = \frac{1}{4}\pi 4^2 = 4\pi$

$$\int_0^4 y\sqrt{16 - y^2} \, dy = \left. \frac{-(16 - y^2)^{3/2}}{3} \right|_0^4 = 0 + \frac{16^{3/2}}{3} = \frac{64}{3}$$

$$F = 12\rho g(4\pi) - 2\rho g\left(\frac{64}{3}\right) = \rho g\left(48\pi - \frac{128}{3}\right)$$

16. Find the interval of convergence of the series $\sum_{n=2}^{\infty} \frac{(x - 3)^n}{n(\ln n)^2}$.

Be sure to check the endpoints.
Name or quote the test(s) you use and check out all requirements of the test.

Ratio Test: $a_n = \frac{(x - 3)^n}{n(\ln n)^2}$ $\quad a_{n+1} = \frac{(x - 3)^{n+1}}{(n + 1)(\ln(n + 1))^2}$

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{|x - 3|^{n+1}}{(n + 1)(\ln(n + 1))^2} \frac{n(\ln n)^2}{|x - 3|^n} = |x - 3| \lim_{n\to\infty} \frac{n}{n + 1} \left(\lim_{n\to\infty} \frac{\ln n}{\ln(n + 1)} \right)^2$$

$$= |x - 3| < 1$$

Converges on $2 < x < 4$. Check endpoints:

At $x = 2$: $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ converges because it is an alternating, decreasing series and $\lim_{n\to\infty} \frac{1}{n(\ln n)^2} = 0$.

At $x = 4$: $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Apply the integral test: Let $u = \ln n$. Then $du = \frac{1}{n} dn$ and

$\int_2^{\infty} \frac{1}{n(\ln n)^2} dn = \int \frac{1}{u^2} du = \frac{-1}{u} = \left. \frac{-1}{\ln n} \right|_2^{\infty} = \frac{1}{\ln 2}$ which is finite. So $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.

So the interval of convergence is $2 \le x \le 4$.