Spring 2006 Math 152
Exam 1A: Solutions
Mon, 20/Feb ©2006, Art Belmonte

1. (c) We have
$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{4-1}\int_1^4 \left(x^2-1\right)^{1/2} x\,dx$
$= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(x^2-1\right)^{3/2}\Big|_1^4 = \frac{1}{9}\cdot 15\sqrt{15} - 0 = \frac{5}{3}\sqrt{15}.$

2. (d) Use integration by parts. First compute an antiderivative, then apply the FTC.

   - Let $\begin{array}{ll} u = x & dv = e^{-2x}\,dx \\ du = dx & v = -\frac{1}{2}e^{-2x} \end{array}$. Then
   $\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}\,dx$
   $= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} = -\frac{1}{4}(2x+1)e^{-2x}.$
   - Hence $\int_0^1 xe^{-2x}\,dx = \left(-\frac{1}{4}(2x+1)e^{-2x}\right)\Big|_0^1$
   $= \left(-\frac{3}{4}e^{-2}\right) - \left(-\frac{1}{4}\right) = \frac{1-3e^{-2}}{4}.$

3. (b) Use trigonometric substitution. Let $x = 5\sin\theta$. Then $dx = 5\cos\theta\,d\theta$ and we have the table

x	0	5
θ	0	$\pi/2$

So

$$\begin{aligned}\int_0^5 \sqrt{25-x^2}\,dx &= \int_0^{\pi/2} 5\cos\theta\cdot 5\cos\theta\,d\theta \\ &= \frac{25}{2}\int_0^{\pi/2} 1+\cos 2\theta\,d\theta \\ &= \frac{25}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2} \\ &= \frac{25}{4}\pi - 0 = \frac{25}{4}\pi.\end{aligned}$$

[*Alternatively*, the integral $\int_0^5 \sqrt{25-x^2}\,dx$ represents the area in the first quadrant under the curve $y = \sqrt{25-x^2}$, part of the circle $x^2+y^2 = 25 = 5^2$. This quarter-circular area is $\frac{1}{4}\pi r^2 = \frac{1}{4}\pi(5)^2 = \frac{25}{4}\pi$.]

4. (d) We'll integrate the rational function via partial fractions.

   - Split the integrand into a sum of partial fractions.

$$\begin{aligned}\frac{1}{x(x-1)(x+1)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \\ 1 &= A\left(x^2-1\right) + B\left(x^2+x\right) + C\left(x^2-x\right) \\ 0x^2+0x+1 &= (A+B+C)x^2 + (B-C)x + -A\end{aligned}$$

   - Equate coefficients of like terms. Thus $1 = -A$, whence $A = -1$. Next $B - C = 0$ implies $C = B$. Substituting for A and C in $A+B+C = 0$ yields $2B - 1 = 0$, whence $B = \frac{1}{2} = C$. Therefore,

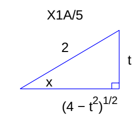
$$\frac{1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}.$$

   - Integrate term-by-term. Recall that $x > 1$. Hence

$$\begin{aligned}&\int \frac{1}{x(x-1)(x+1)}\,dx \\ =& \int \frac{-1}{x} + \frac{\frac{1}{2}}{x-1} + \frac{\frac{1}{2}}{x+1}\,dx \\ =& -\ln x + \frac{1}{2}\ln(x-1) + \frac{1}{2}\ln(x+1) + C \\ =& \ln\left(\frac{\sqrt{x^2-1}}{x}\right) + C\end{aligned}$$

   via the properties of logarithms.

5. (a) If $x = \sin^{-1}\frac{t}{2}$, then x is the angle whose sine (opp / hyp) is $t/2$. Draw a right triangle. Then $\sec x = 1/\cos x$ (hyp/adj)



equals $\dfrac{2}{\sqrt{4-t^2}}$.

6. (b) When the curves $y = x^2$ and $y = \sqrt{x}$ intersect, their y-coordinates are equal. Thus $x^2 = \sqrt{x}$ implies $x^4 = x$. Hence $0 = x^4 - x = x\left(x^3-1\right)$ whence $x = 0, 1$. Since $\left(\frac{1}{4}\right)^2 = \frac{1}{16} < \frac{1}{2} = \sqrt{\frac{1}{4}}$, we conclude that $y = x^2$ lies below $y = \sqrt{x}$ on $[0, 1]$. Therefore the area of the region is given by $\int_0^1 \sqrt{x} - x^2\,dx$.

7. (e) The volume by slicing is $V = \int A(x)\,dx = \int y^2\,dx$
$= \int_{-3}^3 9 - x^2\,dx = 2\int_0^3 9 - x^2\,dx = 2\left(9x - \frac{1}{3}x^3\right)\Big|_0^3$
$= 2(27-9) - 0 = 36.$

8. (c) Use integration by parts. First compute an antiderivative, then apply the FTC.

   - Let $\begin{array}{ll} u = \ln(2x) & dv = dx \\ du = \frac{2}{2x}\,dx = \frac{1}{x}\,dx & v = x \end{array}$. Then
   $\int \ln(2x)\,dx = x\ln(2x) - \int 1\,dx$
   $= x\ln(2x) - x = x(\ln(2x) - 1).$
   - Hence $\int_1^e \ln(2x)\,dx = x(\ln(2x)-1)\Big|_1^e$
   $= (e(\ln(2e)-1)) - (\ln 2 - 1) =$
   $e(\ln 2 + 1 - 1) - \ln 2 + 1 = e\ln 2 - \ln 2 + 1.$

9. (a) This is a trigonometric integral. First compute an antiderivative, then apply the FTC.

$$\begin{aligned}\int (\sin 2x)^3\,dx &= \int \sin 2x\left(1-\cos^2 2x\right)\,dx \\ &= \int \sin 2x\,dx + \int (\cos 2x)^2(-\sin 2x)\,dx \\ &= -\frac{1}{2}\cos 2x + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)(\cos 2x)^3\end{aligned}$$

Therefore, $\int_0^{\pi/2} (\sin 2x)^3\,dx = \left(\frac{1}{6}\cos^3 2x - \frac{1}{2}\cos 2x\right)\Big|_0^{\pi/2}$
$= \left(-\frac{1}{6} + \frac{1}{2}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) = 1 - \frac{1}{3} = \frac{2}{3}.$

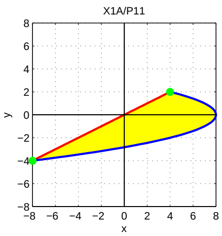
10. (b) Via Hooke's Law we have $F(x) = kx$ or $12 = 2k$, whence $k = 6$. The work done is

$$W = \int_a^b F(x)\,dx = \int_0^4 6x\,dx = 3x^2\Big|_0^4 = 48 \text{ J}.$$

11. When the curves $x = 2y$ and $x = 8 - y^2$ intersect, their x-coordinates are equal. Thus $2y = 8 - y^2$ implies $0 = y^2 + 2y - 8 = (y+4)(y-2)$ whence $y = -4, 2$. Since $2(0) = 0 < 8 = 8 - 0^2$, we conclude that $x = 2y$ lies to the left of $x = 8 - y^2$ on $[-4, 2]$. The area of the region is given by $\int_{-4}^{2} 8 - y^2 - 2y\,dy$, which we now compute.

$$\begin{aligned} &= \left(8y - \tfrac{1}{3}y^3 - y^2\right)\Big|_{-4}^{2} \\ &= \left(16 - \tfrac{8}{3} - 4\right) - \left(-32 + \tfrac{64}{3} - 16\right) \\ &= 12 - \tfrac{8}{3} + 48 - \tfrac{64}{3} \\ &= 60 - \tfrac{72}{3} = 60 - 24 = 36 \end{aligned}$$

Here is a picture of the region.



12. (a) Let $3x = 2\sec\theta$. Then $3\,dx = 2\sec\theta\tan\theta\,d\theta$ or $dx = \frac{2}{3}\sec\theta\tan\theta\,d\theta$. Hence (pic at bottom right!)

$$\begin{aligned} \int \frac{1}{\sqrt{9x^2-4}}\,dx &= \int \frac{\frac{2}{3}\sec\theta\tan\theta\,d\theta}{2\tan\theta} \\ &= \tfrac{1}{3}\int \sec\theta\,d\theta \\ &= \tfrac{1}{3}\ln|\sec\theta + \tan\theta| + C \\ &= \frac{1}{3}\ln\left|\frac{3x}{2} + \frac{\sqrt{9x^2-4}}{2}\right| + C \end{aligned}$$

or $\frac{1}{3}\ln\left|3x + \sqrt{9x^2-4}\right| + K$ via log properties.

(b) Let $u = x^5$. Then $du = 5x^4\,dx$ or $\frac{1}{5}\,du = x^4\,dx$. Thus

$$\begin{aligned} \int \frac{x^4}{\sqrt{1-x^{10}}}\,dx &= \int \frac{x^4}{\sqrt{1-(x^5)^2}}\,dx \\ &= \frac{1}{5}\int \frac{1}{\sqrt{1-u^2}}\,du \\ &= \tfrac{1}{5}\sin^{-1}u + C \\ &= \tfrac{1}{5}\sin^{-1}(x^5) + C. \end{aligned}$$

(c) Use integration by parts.

Let $\begin{array}{ll} u = \tan^{-1}x & dv = x\,dx \\ du = \frac{1}{1+x^2}\,dx & v = \frac{1}{2}x^2 \end{array}$. Then

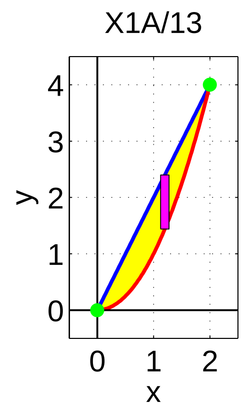
$$\begin{aligned} \int x\tan^{-1}x\,dx &= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx \\ &= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\,dx \\ &= \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C \end{aligned}$$

or $\dfrac{(x^2+1)\tan^{-1}x - x}{2} + C$.

13. When the curves $y = x^2$ and $y = 2x$ intersect, their y-coordinates are equal. Thus $x^2 = 2x$ implies $0 = x^2 - 2x = x(x-2)$ whence $x = 0, 2$. Since $1^2 = 1 < 2 = 2(1)$, we conclude that $y = x^2$ lies below $y = 2x$ on $[0, 2]$. Using washers, the volume swept out by revolving the region between these curves about the x-axis is given by $\int_a^b \pi r_o^2 - \pi r_i^2\,dx = \pi\int_0^2 (2x)^2 - \left(x^2\right)^2\,dx$, which we now compute.

$$\begin{aligned} \pi\int_0^2 4x^2 - x^4\,dx &= \pi\left(\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right)\Big|_0^2 \\ &= \pi\left(\tfrac{32}{3} - \tfrac{32}{5}\right) - 0 \\ &= 32\pi\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = 32\pi\left(\tfrac{5-3}{15}\right) = \frac{64\pi}{15} \end{aligned}$$

Here is a figure of the region that is rotated about the x-axis.



14. When the curves $x = y^2$ and $x = y^{1/3}$ intersect, their x-coordinates are equal. Thus $y^2 = y^{1/3}$ implies $0 = y^6 - y = y\left(y^5 - 1\right)$ whence $y = 0, 1$. Since $\left(\frac{1}{8}\right)^2 = \frac{1}{64} < \frac{1}{2} = \left(\frac{1}{8}\right)^{1/3}$, we conclude that $x = y^2$ lies to the left of $x = y^{1/3}$ on $[0, 1]$. Using cylindrical shells, the volume swept out by revolving the region between these curves about the x-axis is given by $\int_c^d 2\pi r w\,dy$ $= 2\pi\int_0^1 y\left(y^{1/3} - y^2\right)\,dy$, which we now compute.

$$\begin{aligned} 2\pi\int_0^1 y^{4/3} - y^3\,dy &= 2\pi\left(\tfrac{3}{7}y^{7/3} - \tfrac{1}{4}y^4\right)\Big|_0^1 \\ &= 2\pi\left(\tfrac{3}{7} - \tfrac{1}{4}\right) - 0 \\ &= 2\pi\left(\tfrac{12-7}{28}\right) = \frac{5\pi}{14} \end{aligned}$$

Here is a figure of the region that is rotated about the x-axis.