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MATH 152 Honors   Final Exam   Fall 2007

Sections 201,202   Solutions   P. Yasskin

Multiple Choice: (6 points each)

1. A 2 meter bar has linear density $\rho = 1 + x^3$ kg/m where x is measured from one end. Find the average density of the bar.

- **a.** 2 kg/m
- **b.** 3 kg/m   correct choice
- **c.** 4.5 kg/m
- **d.** 5 kg/m
- **e.** 6 kg/m

$$\rho_{\text{ave}} = \frac{1}{2}\int_0^2 (1 + x^3)\,dx = \frac{1}{2}\left[x + \frac{x^4}{4}\right]_0^2 = \frac{1}{2}(2+4) = 3$$

2. A 2 meter bar has linear density $\rho = 1 + x^3$ kg/m where x is measured from one end. Find the center of mass of the bar.

- **a.** $\frac{5}{7}$ m
- **b.** $\frac{5}{6}$ m
- **c.** $\frac{6}{5}$ m
- **d.** $\frac{7}{5}$ m   correct choice
- **e.** $\frac{42}{5}$ m

$$M = \int_0^2 (1 + x^3)\,dx = \left[x + \frac{x^4}{4}\right]_0^2 = 2 + 4 = 6$$

$$M_1 = \int_0^2 x(1 + x^3)\,dx = \left[\frac{x^2}{2} + \frac{x^5}{5}\right]_0^2 = \frac{42}{5}$$

$$\bar{x} = \frac{M_1}{M} = \frac{42}{5 \cdot 6} = \frac{7}{5}$$

3. Compute $\int x \arctan x \, dx$.

a. $\frac{3x^2}{2} \arctan x - \frac{x}{2} \ln(x^2+1) + C$

b. $\frac{3x^2}{2} \arctan x + \frac{x}{2} \ln(x^2+1) + C$

c. $\frac{x^2}{2} \arctan x + \frac{3x}{2} \ln(x^2+1) + C$

d. $\frac{x^2}{2} \arctan x - \frac{1}{2} x - \frac{1}{2} \arctan x + C$

e. $\frac{x^2}{2} \arctan x - \frac{1}{2} x + \frac{1}{2} \arctan x + C$ correct choice

$u = \arctan x \qquad dv = x\,dx$

$du = \frac{1}{1+x^2} dx \qquad v = \frac{x^2}{2}$

$$\int x \arctan x \, dx = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$$

$$\int \frac{x^2}{1+x^2} dx = \int \frac{x^2+1-1}{1+x^2} dx = \int 1 - \frac{1}{1+x^2} dx = x - \arctan x + C$$

$$\int x \arctan x \, dx = \frac{x^2}{2} \arctan x - \frac{1}{2}(x - \arctan x) + C$$

(The second integral can also be done with the trig substitution $x = \tan\theta$.)

4. Find the arclength of the parametric curve $x = t^4 \quad y = \frac{1}{2} t^6$ for $0 \le t \le 1$.

a. $\frac{61}{54}$ correct choice

b. $\frac{16}{9}$

c. $\frac{11}{9}$

d. $\frac{1}{9}$

e. $\frac{1}{54}$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_0^1 \sqrt{(4t^3)^2 + (3t^5)^2} \, dt = \int_0^1 \sqrt{16t^6 + 9t^{10}} \, dt = \int_0^1 t^3 \sqrt{16+9t^4} \, dt$$

$u = 16 + 9t^4 \qquad du = 36t^3\,dt \qquad \frac{1}{36} du = t^3\,dt$

$$L = \frac{1}{36} \int_{16}^{25} \sqrt{u} \, du = \frac{1}{36}\left[\frac{2u^{3/2}}{3}\right]_{16}^{25} = \frac{1}{54}(25^{3/2} - 16^{3/2}) = \frac{1}{54}(125-64) = \frac{61}{54}$$

5. Which term appears in the partial fraction expansion of $\dfrac{4x^2 - 2x + 2}{(x-1)^2(x^2+1)}$?

a. $\dfrac{-2}{(x-1)^2}$

b. $\dfrac{1}{(x-1)^2}$

c. $\dfrac{2}{(x-1)^2}$ correct choice

d. $\dfrac{-2}{x-1}$

e. $\dfrac{2}{x-1}$

$$\frac{4x^2 - 2x + 2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$$

$$4x^2 - 2x + 2 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

$x = 1$: $\quad 4 - 2 + 2 = 0 + B(2) + 0 \qquad 4 = 2B \qquad B = 2$

6. The base of a solid is the region bounded by the curves $y = x^2$, $y = -x^2$ and $x = 2$. The cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

a. $\dfrac{128}{5}$ correct choice

b. $\dfrac{32}{5}$

c. $\dfrac{16}{3}$

d. $\dfrac{8}{3}$

e. $\dfrac{16}{15}$

This is a x-integral. The side of the square is $s = 2x^2$. So the area is $A = s^2 = 4x^4$.

So the volume is

$$V = \int_0^2 A\,dx = \int_0^2 4x^4\,dx = \left.\frac{4x^5}{5}\right|_0^2 = \frac{128}{5}$$

7. Find the solution of the differential equation $\dfrac{dy}{dx} = xy^2 + x$ satisfying the initial condition $y(0) = 1$.

a. $y = \sin\left(\dfrac{x^2}{2} + \dfrac{\pi}{2}\right)$

b. $y = \tan\left(\dfrac{x^2}{2} + \dfrac{\pi}{4}\right)$ correct choice

c. $y = \sin\left(\dfrac{x^2}{2}\right) + 1$

d. $y = \tan\left(\dfrac{x^2}{2}\right) + 1$

e. $y = \cos\left(\dfrac{x^2}{2}\right)$

Separate variables: $\dfrac{dy}{y^2+1} = x\,dx$ $\displaystyle\int \frac{dy}{y^2+1} = \int x\,dx$ $\arctan y = \dfrac{x^2}{2} + C$

Apply the initial conditions $x = 0$, $y = 1$: $\arctan 1 = C = \dfrac{\pi}{4}$

Substitute back and solve for y: $\arctan y = \dfrac{x^2}{2} + \dfrac{\pi}{4}$ $y = \tan\left(\dfrac{x^2}{2} + \dfrac{\pi}{4}\right)$

8. If $g(x) = \cos(x^2)$, what is $g^{(8)}(0)$, the 8th derivative at zero?
HINT: What is the coefficient of x^8 in the Maclaurin series for $\cos(x^2)$?

a. $8 \cdot 7 \cdot 6 \cdot 5$ correct choice

b. $\dfrac{1}{8 \cdot 7 \cdot 6 \cdot 5}$

c. $4!$

d. $\dfrac{1}{4!}$

e. $\dfrac{1}{8!}$

On the one hand, the Maclaurin series for $\cos(t)$ is $\cos(t) = 1 - \dfrac{t^2}{2!} + \dfrac{t^4}{4!} - \cdots$.

So the Maclaurin series for $\cos(x^2)$ is $\cos(x^2) = 1 - \dfrac{x^4}{2!} + \dfrac{x^8}{4!} - \cdots$.

On the other hand the Maclaurin series for any function $g(x)$ is

$$g(x) = g(0) + g'(0)\,x + \cdots + \frac{g^{(8)}(0)}{8!}\,x^8 + \cdots$$

Since these must be equal, the coefficients of x^8 must be equal: $\dfrac{g^{(8)}(0)}{8!} = \dfrac{1}{4!}$

So $g^{(8)}(0) = \dfrac{8!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$

9. Suppose the series $\sum_{n=1}^{\infty} n e^{\left(-n^2\right)}$ is approximated by its 9^{th} partial sum $\sum_{n=1}^{9} n e^{\left(-n^2\right)}$.
Use an integral to bound the error in this approximation.

a. $\frac{1}{2}e^{-64}$
b. $\frac{1}{2}e^{-81}$ correct choice
c. $\frac{1}{2}e^{-100}$
d. $\frac{1}{2}e^{-121}$
e. $\frac{1}{2}e^{-144}$

The error is $E = \sum_{n=10}^{\infty} n e^{\left(-n^2\right)}$.

So $E \le \int_9^\infty n e^{\left(-n^2\right)}\, dn = -\frac{1}{2} e^{\left(-n^2\right)}\Big|_9^\infty = 0 - -\frac{1}{2} e^{\left(-9^2\right)} = \frac{1}{2} e^{-81}$

10. Find the area of the triangle with vertices $P = (2,-1,3)$, $Q = (1,2,1)$ and $R = (3,1,4)$.

a. $\frac{1}{2}\sqrt{73}$
b. $\sqrt{73}$
c. $\frac{5}{2}\sqrt{3}$ correct choice
d. $5\sqrt{3}$
e. $-5\sqrt{3}$

The edges are $\overrightarrow{PQ} = Q - P = (-1,3,-2)$ $\overrightarrow{PR} = R - P = (1,2,1)$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ -1 & 3 & -2 \\ 1 & 2 & 1 \end{vmatrix} = \hat\imath(3+4) - \hat\jmath(-1+2) + \hat k(-2-3) = (7,-1,-5)$$

$$A = \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \frac{1}{2}\sqrt{7^2+1^2+5^2} = \frac{1}{2}\sqrt{49+1+25} = \frac{1}{2}\sqrt{75} = \frac{5}{2}\sqrt{3}$$

11. If $\vec u$ points North East and $\vec v$ points West, in which direction does $\vec u \times \vec v$ point?

a. North West
b. South
c. South East
d. Up correct choice
e. Down

Hold your right hand with the fingers pointing to the right of forward and the palm facing left.
Then you thumb points up.

Work Out: (Points indicated. Part credit possible.)

12. (12 points) Compute $\int_2^4 \frac{8}{x^3\sqrt{x^2-4}}\,dx$

$x = 2\sec\theta \qquad dx = 2\sec\theta\tan\theta\,d\theta$

$x = 2$ @ $\sec\theta = 1$ or $\theta = 0$ $\qquad$ $x = 4$ @ $\sec\theta = 2$ or $\theta = \frac{\pi}{3}$

$$\int_2^4 \frac{8}{x^3\sqrt{x^2-4}}\,dx = \int_0^{\pi/3} \frac{8 \cdot 2\sec\theta\tan\theta\,d\theta}{8\sec^3\theta\sqrt{4\sec^2\theta-4}} = \int_0^{\pi/3} \frac{\sec\theta}{\sec^3\theta}\,d\theta = \int_0^{\pi/3} \cos^2\theta\,d\theta$$

$$= \int_0^{\pi/3} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/3} = \frac{1}{2}\left[\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right] = \frac{\pi}{6} + \frac{\sqrt{3}}{8}$$

13. (15 points) The curve $y = x^2$ is rotated about the y-axis to form a bowl. If the bowl contains 8π cm^3 of water, what is the height of the water in the bowl?

The silce at height y is a circle. So its area is $A = \pi r^2$.

The radius is $r = x = \sqrt{y}$ So the area is $A = \pi y$

and the volume of the slice of thickness dy is $dV = \pi y\,dy$.

So the volume up to height h is

$$V = \int dV = \int_0^h \pi y\,dy = \pi\left[\frac{y^2}{2}\right]_0^h = \frac{\pi}{2}h^2$$

We equate the volume to 8π and solve for h:

$\frac{\pi}{2}h^2 = 8\pi \quad \Rightarrow \quad h^2 = 16 \quad \Rightarrow \quad h = 4$ cm

14. (10 points) This question is designed to teach you about infinite products.

a. (2 pt) Define the partial sum, S_k, and the sum, S, of an infinite series $\sum_{n=0}^{\infty} a_n$.

(Give one sentence including one equation for each.)

The k^{th}-partial sum of the infinite series $\sum_{n=0}^{\infty} a_n$ is $S_k = \sum_{n=0}^{k} a_n$.

The sum of the infinite series $\sum_{n=0}^{\infty} a_n$ is $S = \lim_{k\to\infty} S_k$.

b. (2 pt) By analogy, define the partial product, P_k, and the product, P, of an infinite product $\prod_{n=0}^{\infty} a_n$. (Give one sentence including one equation for each.)

The k^{th}-partial product of the infinite product $\prod_{n=0}^{\infty} a_n$ is $S_k = \prod_{n=0}^{k} a_n$.

The product of the infinite product $\prod_{n=0}^{\infty} a_n$ is $P = \lim_{k\to\infty} P_k$.

c. (4 pt) Compute $\prod_{n=0}^{\infty}\left(1 + x^{(2^n)}\right)$.

HINT: Multiply out the first 3 partial products P_0, P_1 and P_2. Then find P_k and P.

$$P_0 = (1+x)$$

$$P_1 = (1+x)(1+x^2) = 1 + x + x^2 + x^3$$

$$P_2 = (1+x)(1+x^2)(1+x^4) = (1+x+x^2+x^3)(1+x^4) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7$$

$$P_k = 1 + x + \cdots + x^{\left(2^{k+1}-1\right)} = \sum_{n=0}^{2^{k+1}-1} x^n$$ (This could be proved by mathematical induction.)

$$P = \lim_{k\to\infty} P_k = \lim_{k\to\infty} \sum_{n=0}^{2^{k+1}-1} x^n = \sum_{n=0}^{\infty} x^n$$

d. (2 pt) For which x does this infinite product converge? To what function?

It converges when $|x| < 1$ to $P = \dfrac{1}{1-x}$.