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MATH 152 Honors — Final Exam — Fall 2008

Sections 201,202 — Solutions — P. Yasskin

Multiple Choice: (5 points each)

1. Find the average value of $f(x) = \sin x$ on the interval $0 \le x \le \pi$.

a. $\frac{1}{\pi}$

b. $\frac{2}{\pi}$ correct choice

c. $\frac{3}{\pi}$

d. $\frac{1}{2}$

e. $\frac{1}{3}$

$$f_{ave} = \frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{1}{\pi}(-\cos x)\Big|_0^\pi = \frac{1}{\pi}(--1--1) = \frac{2}{\pi}$$

2. Compute $\int_0^3 \frac{1}{(25-x^2)^{3/2}}\,dx$.

a. $\frac{4}{15}$

b. $\frac{4}{75}$

c. $\frac{3}{20}$

d. $\frac{3}{100}$ correct choice

e. $\frac{3}{500}$

Let $x = 5\sin\theta$. Then $dx = 5\cos\theta\,d\theta$.

$$\int \frac{1}{(25-x^2)^{3/2}}\,dx = \int \frac{5\cos\theta}{(25-25\sin^2\theta)^{3/2}}\,d\theta = \frac{1}{25}\int \frac{\cos\theta}{\cos^3\theta}\,d\theta = \frac{1}{25}\int \sec^2\theta\,d\theta = \frac{1}{25}\tan\theta + C$$

$$\sin\theta = \frac{x}{5} \qquad \cos\theta = \sqrt{1-\left(\frac{x}{5}\right)^2} = \frac{\sqrt{25-x^2}}{5} \qquad \tan\theta = \frac{x}{\sqrt{25-x^2}}$$

$$\int_0^3 \frac{1}{(25-x^2)^{3/2}}\,dx = \frac{1}{25}\frac{x}{\sqrt{25-x^2}}\Bigg|_0^3 = \frac{3}{25\cdot 4} = \frac{3}{100}$$

3. The region below $y = x^2$ for $0 \le x \le 2$ is rotated about the y-axis. Find the volume of the solid swept out.

a. 8π correct choice

b. 4π

c. $\dfrac{32\pi}{5}$

d. $\dfrac{32\pi}{3}$

e. $\dfrac{16\pi}{3}$

x-integral cylinders $V = \int_0^2 2\pi r h\,dx = \int_0^2 2\pi x x^2\,dx = 2\pi \dfrac{x^4}{4}\Big|_0^2 = 8\pi$

4. Compute $\int \dfrac{x+2}{x^3+x}\,dx$. HINT: Equate coefficients.

a. $\ln \dfrac{x^2}{(x^2+1)^2} + \arctan(x) + C$

b. $\ln \dfrac{x^2}{(x^2+1)^2} - \arctan(x) + C$

c. $\ln\left|\dfrac{x}{x^2+1}\right| + \arctan(x) + C$

d. $\ln\left|\dfrac{x}{x^2+1}\right| - \arctan(x) + C$

e. $\ln \dfrac{x^2}{x^2+1} + \arctan(x) + C$ correct choice

$\dfrac{x+2}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$ $\quad x+2 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + Cx + A$

$A = 2 \quad C = 1 \quad A+B = 0 \quad B = -2$

$$\int \frac{x+2}{x^3+x}\,dx = \int \frac{2}{x} + \frac{-2x+1}{x^2+1}\,dx = \int \frac{2}{x} - \frac{2x}{x^2+1} + \frac{1}{x^2+1}\,dx$$

$$= 2\ln|x| - \ln(x^2+1) + \arctan(x) + C = \ln\frac{x^2}{x^2+1} + \arctan(x) + C$$

5. Which of the following differential equations is NOT separable?

a. $\dfrac{dy}{dx} = x + xy$

b. $\dfrac{dy}{dx} = xy + xy^2 + x^2y + x^2y^2$

c. $\dfrac{dy}{dx} = \dfrac{1}{xy} + \dfrac{1}{x} + \dfrac{1}{y}$ correct choice

d. $\dfrac{dy}{dx} = \dfrac{x}{y} + x + \dfrac{1}{y} + 1$

e. $\dfrac{dy}{dx} = \dfrac{y}{x} - y$

$x + xy - x(1+y)$ $\quad xy + xy^2 + x^2y + x^2y^2 = (x+x^2)(y+y^2)$

$\dfrac{x}{y} + x + \dfrac{1}{y} + 1 = (x+1)\left(\dfrac{1}{y} + 1\right)$ $\quad \dfrac{y}{x} - y = y\left(\dfrac{1}{x} - 1\right)$

6. Suppose $y = f(x)$ is the solution of the differential equation $\dfrac{dy}{dx} = xy^2 + x^2$ satisfying the initial condition $f(1) = 2$. Find $f'(1)$.

- **a**. 1
- **b**. 2
- **c**. 3
- **d**. 4
- **e**. 5 correct choice

$f'(x) = \dfrac{dy}{dx} = xy^2 + x^2$ When $x = 1$, we have $y = f(1) = 2$. So $f'(1) = (1)(2)^2 + (1)^2 = 5$

7. Compute $\displaystyle\sum_{n=1}^{\infty}(e^{1/n} - e^{1/(n+1)})$.

- **a**. e
- **b**. $1 - e$
- **c**. $e - e^2$
- **d**. $e - 1$ correct choice
- **e**. $e^2 - e$

$$S_k = \sum_{n=1}^{k}(e^{1/n} - e^{1/(n+1)}) = (e - e^{1/2}) + (e^{1/2} - e^{1/3}) + \cdots + (e^{1/k} - e^{1/(k+1)}) = e - e^{1/(k+1)}$$

$$S = \lim_{n\to\infty} S_k = \lim_{n\to\infty}(e - e^{1/(k+1)}) = e - e^0 = e - 1$$

8. The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n} + n}$

- **a**. Converges by the n^{th}-Term Divergence Test.
- **b**. Diverges by the Comparison Test with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$.
- **c**. Diverges by the Comparison Test with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.
- **d**. Diverges by the Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$. correct choice
- **e**. Diverges by the Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

n^{th}-Term Divergence Test fails. $\dfrac{1}{\sqrt{n} + n} < \dfrac{1}{n} \le \dfrac{1}{\sqrt{n}}$ $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n} + n}\frac{\sqrt{n}}{1} = 0$

$\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n} + n}\frac{n}{1} = 1$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series which is divergent.

9. The series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

a. Converges by the Ratio Test.

b. Diverges by the Integral Test. correct choice

c. Converges because it is a p-series with $p = 2 > 1$.

d. Converges because it is a p-series with $p = 1$.

e. Diverges by the Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

$\frac{n}{n^2+1}$ is positive and decreasing, and $\int_1^{\infty} \frac{n}{n^2+1}\, dn = \frac{1}{2}\ln(n^2+1)\Big|_1^{\infty} = \infty$

Ratio Test fails. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is not a p-series. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

10. Compute $\lim_{x\to 0} \frac{\sin(x^2) - x^2}{x^5}$

a. $-\frac{1}{3}$

b. $-\frac{1}{6}$

c. 0 correct choice

d. $\frac{1}{6}$

e. $\frac{1}{3}$

$\sin(t) = t - \frac{t^3}{3!} + \cdots \qquad \sin(x^2) = x^2 - \frac{x^6}{3!} + \cdots \qquad \sin(x^2) - x^2 = -\frac{x^6}{3!} + \cdots$

$$\lim_{x\to 0} \frac{\sin(x^2) - x^2}{x^5} = \lim_{x\to 0} \frac{-\frac{x^6}{3!} + \cdots}{x^5} = \lim_{x\to 0} -\frac{x}{3!} + \cdots = 0$$

11. Find the volume of the parallelepiped with edge vectors

$\vec{a} = (2,1,4)$, $\vec{b} = (0,3,2)$ and $\vec{c} = (-2,1,3)$.

a. 42

b. 34 correct choice

c. 28

d. 21

e. 17

The is the absolute value of the triple product $\vec{a} \times \vec{b} \cdot \vec{c}$ which is a determinant:

$$V = \left|\vec{a} \times \vec{b} \cdot \vec{c}\right| = \left\| \begin{matrix} 2 & 1 & 4 \\ 0 & 3 & 2 \\ -2 & 1 & 3 \end{matrix} \right\| = |2(9-2) - 1(0+4) + 4(0+6)| = 34$$

Work Out: (Points indicated. Part credit possible.)

12. (12 points) The curve $x = 2t^2$, $y = t^3$ for $0 \le t \le 1$ is rotated about the y-axis. Find the surface area of the surface swept out. Don't simplify your numbers.

$\dfrac{dx}{dt} = 4t \qquad \dfrac{dy}{dt} = 3t^2$ The radius of revolution is $r = x = t^2$.

$$A = \int 2\pi r\, ds = \int_0^1 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi 2t^2 \sqrt{16t^2 + 9t^4}\, dt = \int_0^1 4\pi t^3 \sqrt{16 + 9t^2}\, dt$$

$$u = 16 + 9t^2 \qquad du = 18t\,dt \qquad \frac{du}{18} = t\,dt \qquad t^2 = \frac{u-16}{9}$$

$$A = 4\pi \int_{16}^{25} \frac{u-16}{9} \sqrt{u}\, \frac{du}{18} = \frac{2\pi}{81} \int_{16}^{25} (u^{3/2} - 16u^{1/2})\, du = \frac{2\pi}{81} \left[\frac{2u^{5/2}}{5} - \frac{32u^{3/2}}{3} \right]_{16}^{25}$$

$$= \frac{2\pi}{81} \left(\frac{2(5)^5}{5} - \frac{32(5)^3}{3} \right) - \frac{2\pi}{81} \left(\frac{2(4)^5}{5} - \frac{32(4)^3}{3} \right) = \frac{5692}{1215}\pi$$

13. (22 points) A bucket starts out containing 4 liters of salt water with concentration of 60 gm of salt per liter. Salt water is added to the bucket at 3 liters per minute with concentration of 50 gm of salt per liter. The water in the bucket is kept mixed but is leaking out at the rate of 1 liter per minute. (Note the amount of water in the bucket is **not** constant.)

Let $S(t)$ denote the amount of salt in the bucket at time t.

a. (7 pt) Write a differential equation for $S(t)$ and an initial condition for $S(t)$.
HINT: How much water is in the bucket at time t?

The volume of water at time t is $V = (4+2t)$ L.

$$\frac{dS}{dt} = \underbrace{\frac{50\text{ gm}}{\text{L}}\,\frac{3\text{ L}}{\text{min}}}_{\text{IN}} - \underbrace{\frac{S(t)\text{ gm}}{(4+2t)\text{ L}}\,\frac{1\text{ L}}{\text{min}}}_{\text{OUT}} = 150 - \frac{S(t)}{(4+2t)} \qquad S(0) = \frac{60\text{ gm}}{\text{L}}\cdot 4\text{ L} = 240$$

b. (2 pt) Select **one**: The differential equation is

☐ separable ■ linear ☐ both ☐ neither

c. (9 pt) Solve the initial value problem for $S(t)$.

$$\frac{dS}{dt} + \frac{S(t)}{(4+2t)} = 150$$

$$P(t) = \frac{1}{(4+2t)} \qquad \int P(t)\,dt = \int \frac{1}{(4+2t)}\,dt = \frac{1}{2}\ln(4+2t) = \ln\sqrt{4+2t}$$

$$I = e^{\int P(t)\,dt} = e^{\ln\sqrt{4+2t}} = \sqrt{4+2t}$$

$$\sqrt{4+2t}\,\frac{dS}{dt} + \frac{S(t)}{\sqrt{4+2t}} = 150\sqrt{4+2t} \qquad \frac{d}{dt}\left(\sqrt{4+2t}\,S\right) = 150\sqrt{4+2t}$$

$$\sqrt{4+2t}\,S = \int 150\sqrt{4+2t}\,dt = 50(4+2t)^{3/2} + C$$

At $t = 0$, we have $S = 240$. So $\sqrt{4}\,240 = 50(4)^{3/2} + C$ $\qquad C = 480 - 400 = 80$

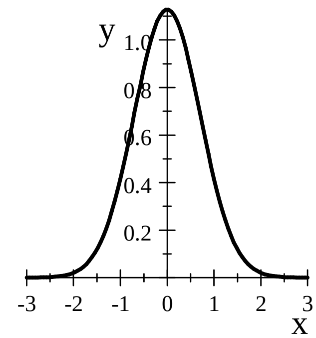
$$\sqrt{4+2t}\,S = 50(4+2t)^{3/2} + 80 \qquad S = 50(4+2t) + \frac{80}{\sqrt{4+2t}} = 200 + 100t + \frac{80}{\sqrt{4+2t}}$$

d. (4 pt) How much salt is in the bucket after 6 minutes?
What is the concentration after 6 minutes?

$$S(6) = 200 + 100\cdot 6 + \frac{80}{\sqrt{4+2\cdot 6}} = 820\text{ gm}$$

$$\text{Concentration} = \frac{S(6)}{V(6)} = \frac{820}{16} = \frac{205}{4} = 51.25\ \frac{\text{gm}}{\text{L}}$$

14. (16 points) The "Gaussian bell curve" used in statistics is $\frac{2}{\sqrt{\pi}} e^{-t^2}$ as shown in the graph. The "error function" is its integral:



$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

which gives the probability that the value of t is between 0 and x. For this problem we will ignore the coefficient and study the function

$$f(x) = \int_0^x e^{-t^2}\,dt$$

a. (6 pt) Find a Maclaurin series for $f(x)$ in summation notation.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

$$f(x) = \int_0^x e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{2n}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$$

b. (6 pt) Use the first 3 terms of the Maclaurin series for $f(x)$ to approximate $f(1)$. $(n = 0, 1, 2)$ Do not simplify. Find a bound on the error in this approximation. Explain why.

$$f(x) \approx \sum_{n=0}^{2} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{10} \qquad f(1) \approx 1 - \frac{1}{3} + \frac{1}{10} \approx 0.76667$$

Since this series is alternating, the error is bounded by the absolute value of the next term.

$$|E| < |a_3| = \left| \frac{(-1)^3 x^7}{3!(7)} \right|_{x=0.1} = \frac{1}{42} =.23810 \times 10^{-1}$$

c. (4 pt) How many terms of the Maclaurin series for $f(x)$ would you need to approximate $f(1)$ to within 10^{-3}? Why?

$$|E| < |a_n| = \left| \frac{(-1)^n x^{2n+1}}{n!(2n+1)} \right|_{x=1} = \frac{1}{n!(2n+1)} < 10^{-3}$$

$n = 4$: $\quad |E| < \frac{1}{4!(9)} = \frac{1}{24 \cdot 9} < 10^{-2}$

$n = 5$: $\quad |E| < \frac{1}{5!(11)} = \frac{1}{120 \cdot 11} < 10^{-3}$ So 5 terms are needed.