Solutions to: Math 152, Fall 2008, Examination I, Version B

1. Evaluate $\int_1^e \frac{\sqrt{\ln x}}{x}\, dx$. Correct answer is (a) 2/3.

Substitute $u = \ln x$. We have $du = \frac{1}{x}\, dx$. When $x = 1$, $u = 0$ and when $x = e$, $u = 1$. The integral becomes

$$\int_1^e (\ln x)^{1/2} \frac{dx}{x} = \int_0^1 u^{1/2} du = [\frac{2}{3} u^{3/2}]_0^1 = \frac{2}{3}.$$

2. Evaluate $\int_0^1 x(2x-1)^7 dx$. Correct answer is (d) 1/18.

Substitute $u = 2x - 1$. We have $x = \frac{1}{2}(u+1)$ and $dx = \frac{1}{2} du$. When $x = 0$, $u = -1$ and when $x = 1$, $u = 1$. The integral becomes

$$\int_{-1}^1 \frac{1}{2}(u+1)u^7 \frac{1}{2} du = \frac{1}{4} \int_{-1}^1 (u^8 + u^7) du$$
$$= \frac{1}{4}[\frac{1}{9} u^9 + \frac{1}{8} u^8]_{-1}^1$$
$$= \frac{1}{4}(\frac{1}{9} - (-1)^9 \frac{1}{9} + \frac{1}{8} - (-1)^8 \frac{1}{8}) = \frac{1}{4} \times \frac{2}{9} = \frac{1}{18}.$$

3. Compute the area under the curve $y = xe^{-x}$, between $x = 0$ and $x = 1$. Correct answer is (a) $1 - 2e^{-1}$.

The area under the curve is the integral

$$A = \int_0^1 xe^{-x} dx.$$

We compute the integral using integration by parts with the choice $u = x$, $dv = e^{-x} dx$. We have $du = dx$ and $v = -e^{-x}$. Applying the integration by parts formula we have

$$A = \int u\, dv = uv - \int v\, du$$
$$= [-xe^{-x}]_0^1 - \int_0^1 (-e^{-x} dx)$$
$$= -e^{-1} + \int_0^1 e^{-x} dx$$
$$= -e^{-1} + [-e^{-x}]_0^1 = -e^{-1} - e^{-1} + 1 = 1 - 2e^{-1}.$$

1

4. Compute the area enclosed by the parabola $y = x^2$ and the straight line $y = 2x$. Correct answer is (e) 4/3.

The parabola and the line intersect when $x^2 = 2x$, so when $x = 0$, $y = 0$ and when $x = 2$, $y = 4$. In the interval $0 \le x \le 2$ we have $2x \ge x^2$. The area between the curves is given by the formula

$$A = \int_a^b |f(x) - g(x)|dx = \int_0^2 (2x - x^2)dx = \left[x^2 - \frac{1}{3}x^3\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}.$$

5. A force of 10lb is required to hold a spring stretched 1/3 ft beyond its natural length. How much work is done in stretching it from its natural length to 1/2 ft beyond its natural length? (Recall and use *Hooke's Law*: the force required to maintain a spring attached x units beyond its natural length is given by $f(x) = kx$, where k is a constant). Correct answer is (a) 15/4 ft-lb.

We first use the information in the first sentence of the problem to compute k. Namely substituting $x = 1/3$ and $f(1/3) = 10$ in Hooke's Law we have $10 = k/3$, so that $k = 30$. The work done in stretching the spring from its natural length to 1/2 ft beyond its natural length is then

$$W = \int f(x)dx = \int_0^{1/2} 30x\,dx = \left[15x^2\right]_0^{1/2} = 15/4.$$

6. Use the trigonometric identity $2 \sin A \cos B \equiv \sin(A+B) + \sin(A-B)$ to compute the integral $\int \sin 3t \cos 2t\, dt$. Correct answer is (a) $-\frac{\cos 5t}{10} - \frac{\cos t}{2} + C$.

Applying the identity with $A = 3t$, $B = 2t$, we have

$$\begin{aligned} \sin 3t \cos 2t &= \frac{1}{2}\left[\sin(3t + 2t) + \sin(3t - 2t)\right] \\ &= \frac{1}{2}\left[\sin 5t + \sin t\right] \end{aligned}$$

Integrating both sides of the above formula we have

$$\int \sin 3t \cos 2t\, dt = \frac{1}{2}\int \sin 5t\, dt + \frac{1}{2}\int \sin t\, dt = -\frac{1}{10}\cos 5t - \frac{1}{2}\cos t + C.$$

7. Compute $\int \tan^3 \theta \sec \theta \, d\theta$. Correct answer is (c) $\frac{1}{3} \sec^3 \theta - \sec \theta + C$.

$$\int \tan^3 \theta \sec \theta \, d\theta = \int \tan^2 \theta \sec \theta \tan \theta \, d\theta$$

$$= \int (\sec^2 \theta - 1) d \sec \theta$$

$$= \frac{1}{3} \sec^3 \theta - \sec \theta + C.$$

For the last step, you may use the substitution $u = \sec \theta$.

8. Compute $\int \frac{dx}{(3-x)(3+x)}$. Correct answer is (d) $-\frac{1}{6} \ln |\frac{x-3}{x+3}|$.

We decompose $\frac{1}{(3-x)(3+x)}$ into partial fractions as follows:

$$\frac{1}{(3-x)(3+x)} = \frac{A}{3-x} + \frac{B}{3+x}.$$

Clearing denominators on both sides (multiplying both sides by $(3-x)(3+x)$) we have

$$1 = A(3+x) + B(3-x) = (A-B)x + 3A + 3B.$$

Equating the coefficients of x in this equation gives the two equations $A - B = 0$ and $3A + 3B = 1$ whose solution is $A = B = 1/6$. Substituting back the values of A and B into the partial fraction decomposition gives

$$\frac{1}{(3-x)(3+x)} = \frac{1/6}{3-x} + \frac{1/6}{3+x}.$$

Integrating both sides of this last equation yields

$$\int \frac{1}{(3-x)(3+x)} \, dx = \frac{1}{6} \int \frac{1}{3-x} \, dx + \frac{1}{6} \int \frac{1}{3+x} \, dx$$

$$= -\frac{1}{6} \int \frac{1}{x-3} \, dx + \frac{1}{6} \int \frac{1}{x+3} \, dx$$

$$= -\frac{1}{6} \ln |x-3| + \frac{1}{6} \ln |x+3| + C = -\frac{1}{6} \ln |\frac{x-3}{x+3}|.$$

9. Which of the following is the correct partial fraction decomposition for the rational function
$$\frac{x^4 + x + 1}{(x^2 - 1)^2(x^2 + 1)^2}?$$

The correct answer is (e)

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2}.$$

First observe that the degree of the numerator of the rational function is strictly less than the degree of the denominator. Therefore, the rational function is proper and the form of its decomposition into partial fractions can be seen from the factorization of its denominator. We have, for this denominator,

$$(x^2 - 1)^2(x^2 + 1)^2 = ((x - 1)(x + 1))^2(x^2 + 1)^2 = (x - 1)^2(x + 1)^2(x^2 + 1)^2.$$

On the right hand side of the above equation we have the factorization into distinct linear and quadratic factors (with repeated factors being expressed as powers of a single factor). The quadratic factor $x^2 + 1$ cannot be further factorized. The answer now follows from the theorem on partial fraction decompositions given in Stewart: it is a combination of his Case II and Case IV in §8.4.

10. Suppose that f is a continuous function defined on $[0, \infty)$ and that the average value of f over the interval $[0, t]$ equals $t^2 + 1$ for every $t > 0$. Find f. Correct answer is (e) $f(x) = 3x^2 + 1$.

The average value f_{ave} of a function over the interval $[a, b]$ is given by the formula

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx.$$

Applying this with $a = 0$ and $b = t$ we have

$$f_{\text{ave}} = f_{\text{ave}}(t) = \frac{1}{t} \int_0^t f(x) dx.$$

We are told in the problem statement that $f_{\text{ave}}(t) = t^2 + 1$. Therefore

$$f_{\text{ave}}(t) = \frac{1}{t} \int_0^t f(x) dx = t^2 + 1.$$

Multiplying both sides of the above equation by t we have

$$\int_0^t f(x) dx = t^3 + t.$$

4

We now differentiate both sides and use the Fundamental Theorem of Calculus (which tells us that differentiation and integration are inverses of one another) to deduce

$$\frac{d}{dt}\int_0^t f(x)dx = f(t) = 3t^2 + 1.$$

Writing the variable as x instead of t we have $f(x) = 3x^2 + 1$.

11. Let $a > 0$, and let $\mathcal{R}$ be the region enclosed by $y = 2x^3$, the x-axis, and the line $x = a$. Let $\mathcal{S}$ denote the solid obtained by rotating $\mathcal{R}$ about the x-axis. Ascertain the value of a for which the volume of $\mathcal{S}$ is π times the area of $\mathcal{R}$. The answer is $a = 7^{1/3}/2$.

The area A of $\mathcal{R}$ is given by

$$A = \int_0^a 2x^3 dx = [\frac{1}{2}x^4]_0^a = \frac{1}{2}a^4.$$

We compute the volume V of $\mathcal{S}$ using the disk method. We have

$$V = \int \pi y^2 dx = \int_0^a \pi(2x^3)^2 dx = \pi \int_0^a 4x^6 dx = \pi[\frac{4}{7}x^7]_0^a = \frac{4}{7}\pi a^7.$$

We want to find a such that $V = \pi A$. Substituting in the formulae for V and A this means that we want to find a such that

$$V = \frac{4}{7}\pi a^7 = \pi A = \frac{1}{2}\pi a^4.$$

Dividing both sides by $4\pi a^4/7$ we have $a^3 = 7/8$, so that $a = 7^{1/3}/2$.

12. Let $\mathcal{S}$ be the solid whose base is the triangle with vertices $(0,0)$, $(1,0)$, and $(0,2)$, and whose cross sections perpendicular to the y-axis are semicircles. Compute the volume of $\mathcal{S}$. The answer is $\pi/12$.

The triangle has sides along the x-axis, the y-axis and the line $x = \frac{1}{2}(2 - y)$. The cross section at height y is a semicircle of diameter $x = \frac{1}{2}(2 - y)$. Therefore the area of this cross section is the area of half a circle of radius $r = x/2$, namely

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(\frac{x}{2})^2 = \frac{1}{2}\pi(\frac{1}{4}(2 - y))^2 = \frac{1}{32}\pi(4 - 4y + y^2).$$

The volume of $\mathcal{S}$ is given by

$$V = \int Ady = \int_0^2 \frac{1}{32}\pi(4 - 4y + y^2)dy = \frac{\pi}{32}[4y - 2y^2 + \frac{1}{3}y^3]_0^2 = \frac{\pi}{32}[8 - 8 + \frac{8}{3}] = \frac{\pi}{12}.$$

13. (i) Let $\mathcal{R}_t$ denote the region enclosed by the y-axis, the line $y = 1$, and the curve $y = \sqrt{x}$. Use the method of disks to compute the volume of the solid obtained by rotating the region $\mathcal{R}_t$ about the y-axis. The answer is $\pi/5$.

The volume of the solid is given by

$$V = \int \pi x^2 dy = \int_0^1 \pi (y^2)^2 dy = \int_0^1 \pi y^4 dy = \pi \left[\frac{1}{5}y^5\right]_0^1 = \pi/5.$$

(ii) Let $\mathcal{R}_b$ denote the region enclosed by the x-axis, the line $x = 1$, and the curve $y = \sqrt{x}$. Use the method of cylindrical shells to calculate the volume of the solid obtained by rotating $\mathcal{R}_b$ about the line $x = 1$. The answer is $8\pi/15$.

The volume of the solid is given by $V = \int 2\pi r h\, dr$. Here $dr = dx$ and r is the radius of the cylindrical shell at distance x from the y-axis. As we are rotating about the line $x = 1$, this radius is the distance from x to the line $x = 1$, so is given by $r = 1 - x$. The height h is given by the value of y at x, so equals $y = \sqrt{x}$. Therefore our integral becomes

$$V = \int_0^1 2\pi(1-x)\sqrt{x}\,dx = 2\pi\left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1 = 4\pi\left(\frac{1}{3} - \frac{1}{5}\right) = 8\pi/15.$$

14. Compute each of the following integrals

(i) $\int \frac{(x-1)^2}{x^3+x}\,dx$. The answer is $\ln|x| - 2\tan^{-1}(x) + C$.

The rational function in the integrand is proper (the degree of the numerator is strictly less than the degree of the denominator). The denominator factorizes completely as follows:

$$x^3 + x = x(x^2 + 1).$$

Therefore the partial fraction decomposition of the rational function is of the form

$$\frac{(x-1)^2}{x^3+x} = \frac{x^2 - 2x + 1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}.$$

Clearing denominators (multiplying both sides by $x(x^2+1)$) we have

$$x^2 - 2x + 1 = A(x^2 + 1) + (Bx + C)x = (A + B)x^2 + Cx + A.$$

Equating coefficients of powers of x gives $A = 1$, $C = -2$ and $A + B = 1$, so that $B = 0$. Substituting back into the partial fraction decomposition we have

$$\frac{(x-1)^2}{x^3+x} = \frac{1}{x} + \frac{-2}{x^2 + 1}.$$

6

Integrating both sides of this last equation gives

$$\int \frac{(x-1)^2}{x^3+x} \, dx = \int \frac{1}{x} \, dx - 2 \int \frac{1}{x^2+1} \, dx = \ln|x| - 2\tan^{-1} x + C.$$

(ii) $\int 4x \tan^{-1}(2x) dx$. The answer is

$$\frac{1}{2}(1+4x^2)\tan^{-1}(2x) - x + C.$$

We use integration by parts. First we write $\int 4x \tan^{-1}(2x) dx$ in the form $\int u \, dv$ by setting $u = \tan^{-1}(2x)$ and $dv = 4x \, dx$. Then, $du = \frac{2}{1+4x^2}$ and $v = 2x^2$. The integration by parts formula gives

$$\int 4x \tan^{-1}(2x) dx = \int u \, dv = uv - \int v \, du$$

$$= 2x^2 \tan^{-1}(2x) - \int \frac{4x^2}{1+4x^2} dx$$

$$= 2x^2 \tan^{-1}(2x) - \int \frac{1+4x^2-1}{1+4x^2} dx$$

$$= 2x^2 \tan^{-1}(2x) - \int \left(1 - \frac{1}{1+4x^2}\right) dx$$

$$= 2x^2 \tan^{-1}(2x) - \int dx + \int \frac{1}{1+4x^2} dx$$

$$= 2x^2 \tan^{-1}(2x) - x + \frac{1}{2}\tan^{-1}(2x) + C$$

$$= \frac{1}{2}(1+4x^2)\tan^{-1}(2x) - x + C.$$

(iii) $\int x\sqrt{1-9x^2} dx$. The answer is

$$\frac{-1}{27}(1-9x^2)^{3/2} + C.$$

Method 1: Make the substitution $u = 1 - 9x^2$. Then $du = -18x \, dx$ and the integral becomes

$$\int \sqrt{u}\left(\frac{-du}{18}\right) = \left(\frac{-1}{18}\right)\left(\frac{2}{3}\right)u^{3/2} + C = \frac{-1}{27}u^{3/2} + C = \frac{-1}{27}(1-9x^2)^{3/2} + C.$$

7

Method 2: Make the substitution $x = \frac{1}{3}\sin\theta$. Then $dx = \frac{1}{3}\cos\theta d\theta$ and $\sqrt{1 - 9x^2} = \sqrt{1 - \sin^2\theta} = \cos\theta$. Therfore the integral becomes

$$
\begin{aligned}
\int x\sqrt{1 - 9x^2}dx &= \int \frac{1}{3}\sin\theta\cos\theta\frac{1}{3}\cos\theta d\theta \\
&= \frac{1}{9}\int \cos^2\theta\sin\theta d\theta \\
&= \frac{-1}{9}\int \cos^2\theta d\cos\theta \\
&= \frac{-1}{27}\cos^3\theta + C \\
&= \frac{-1}{27}(\cos^2\theta)^{3/2} + C \\
&= \frac{-1}{27}(1 - \sin^2\theta)^{3/2} + C \\
&= \frac{-1}{27}(1 - 9x^2)^{3/2} + C.
\end{aligned}
$$