Solutions to MATH 152 Fall 2008 Exam 3A

1. **B** $\lim_{n\to\infty}(a_n^2 - 3b_n) = (\lim_{n\to\infty} a_n)^2 - 3\lim_{n\to\infty} b_n = 2^2 - 3(-3) = 13.$

2. **C** Apply L'Hospital's Rule to $\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0$

3. **D** Complete the square: $x^2 + (y^2 - 2y + 1) + z^2 = 1 + 1$; $x^2 + (y-1)^2 + z^2 = 2$, so $r^2 = 2$ and $r = \sqrt{2}$.

4. **C** Using the Comparison Test. (D) is NOT necessarily true because $\lim_{n\to\infty} a_n = 0$ does not necessarily mean that $\sum_{n=1}^{\infty} a_n$ is convergent.

5. **B** Since $\lim_{n\to\infty}\frac{n}{n+1} = 1$, the sequence $(-1)^n\frac{n}{n+1}$ alternates between -1 and 1, therefore the terms of the series do not approach 0, which means the series diverges by the Test for Divergence.

6. **B** Let $a_n = \frac{1}{n}$. Then $\lim_{n\to\infty}\frac{b_n}{a_n} = 1$, which means the series $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ either both converge or both diverge. Since $\sum_{n=1}^{\infty} a_n$ diverges (by Integral Test or P-Test), $\sum_{n=1}^{\infty} b_n$ diverges by the Limit Comparison Test.

7. **A** $\sum_{n=1}^{\infty}\frac{1}{(n+2)(n+3)} = \sum_{n=1}^{\infty}\left(\frac{1}{n+2} - \frac{1}{n+3}\right)$, which is a Telescoping Series: $s_N = \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{6}\right.$ as $N \to \infty$.

8. **C** Since the terms of both series approach zero, both series converge by the Alternating Series Test. To test absolute convergence, we look at (IA)$\sum_{n=1}^{\infty}\frac{1}{n^{3/4}}$ and (IIA)$\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}$. Both are P-series; in (IA), $P < 1$so the series diverges, and in (IIA), $P > 1$ so the series converges. Therefore, series (I) converges but not absolutely, and series (II) converges absolutely.

9. **B** The series can be written as $\sum_{n=1}^{\infty}\left(\frac{4}{3}\right)\left(\frac{2}{3}\right)^{n-1}$, which is a geometric series with $a = \frac{4}{3}$ and $r = \frac{2}{3}$. The sum of the series is $\frac{\frac{4}{3}}{1-\frac{2}{3}} = 4$.

10. **C** The Maclaurin series for $\cos x$ is $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$. So the Maclaurin series for $\cos(x^2)$ is $\sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n}}{(2n)!}$.

11. (a) $\overline{\mathbf{AB}} =< 1,2,2 >$, $\overline{\mathbf{BC}} =< 0,-1,1 >$. $\overline{\mathbf{AB}}\cdot\overline{\mathbf{BC}} = (1)(0)+(2)(-1)+(2)(1) = 0$, so the sides are perpendicular to each other.

    (b) $|\overline{\mathbf{AB}}| = \sqrt{1^2+2^2+2^2} = 3$, $|\overline{\mathbf{BC}}| = \sqrt{0^2+(-1)^2+1^2} = \sqrt{2}$, so the area is $\frac{1}{2}|\overline{\mathbf{AB}}||\overline{\mathbf{BC}}| = \frac{3\sqrt{2}}{2}$.

12. $f(-1) = 4$; $f'(x) = 8x^3 - 1$, so $f'(-1) = -9$; $f''(x) = 24x^2$, so $f''(-1) = 24$; $f'''(x) = 48x$, so $f'''(-1) = -48$. The third degree Taylor Polynomial is $f(-1) + \frac{f'(-1)}{1!}(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \frac{f'''(-1)}{3!}(x+1)^3 = 4 - 9(x+1) + 12(x+1)^2 - 8(x+1)^3$

13. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$. Therefore, subtracting the first term of the series gives us $e^{-1} - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$.

14. (a) Applying the Ratio Test gives us absolute convergence when $\lim_{n\to\infty} \left| \frac{\frac{(x-1)^{n+1}}{2^{n+1}\sqrt{n+1}}}{\frac{(x-1)^n}{2^n\sqrt{n}}} \right| = \lim_{n\to\infty} \frac{|x-1|}{2} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} < 1$ and divergence when the limit is > 1. Since the second fraction approaches 1, we have absolute convergence when $\frac{|x-1|}{2} < 1$, $|x-1| < 2$ which makes the radius of convergence 2.

(b) The series converges when $-2 < x-1 < 2$, $-1 < x < 3$. To find the interval of convergence, test the endpoints: When $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-2)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series test. Whne $x = 3$, the series becomes $\sum_{n=1}^{\infty} \frac{2^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which diverges by the P-test or integral test. Therefore, the interval of convergence is $-1 \le x < 3$.

15. (a) $\int S(x)\,dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n}\,dx = \sum_{n=1}^{\infty} \int \frac{(-1)^{n+1}}{(2n+1)!} x^{2n}\,dx = C + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(2n+1)!} x^{2n+1}$

(b) $\int_0^{1/2} S(x)\,dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(2n+1)!} x^{2n+1}\Big|_0^{1/2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)(2n+1)!} \left(\frac{1}{2}\right)^{2n+1}$.

(c) Since the series is alternating $|S - S_3| \le |a_4| = \frac{1}{(2\cdot 4+1)(2\cdot 4+1)!} \left(\frac{1}{2}\right)^{2\cdot 4+1} = \frac{1}{9\cdot 9!}\left(\frac{1}{2}\right)^9$.