MATH 152, SPRING 2012
HONORS EXAM II - SOLUTIONS

Last Name: ______________________________ **First Name**: ______________________________

Signature: ______________________________ **Section No**: _______________

PART I: **Multiple Choice** (**4 pts each**)

1. Compute $\int_{-1}^{\infty} \frac{dx}{1+x^2}$.

a. $\frac{3\pi}{4}$ Correct Choice
b. $\frac{\pi}{2}$
c. $\frac{\pi}{4}$
d. ∞
e. 0

Solution: $\int_{-1}^{\infty} \frac{dx}{1+x^2} = \arctan(x)\Big|_{-1}^{\infty} = \frac{\pi}{2} - -\frac{\pi}{4} = \frac{3\pi}{4}$

2. The improper integral $\int_{1}^{e} \frac{dx}{x\ln x}$

a. diverges to ∞. Correct Choice
b. diverges to $-\infty$.
c. converges to 1.
d. converges to -1.
e. converges to $\frac{1}{e} - 1$.

Solution: $u = \ln x \qquad du = \frac{dx}{x} \qquad \int_{1}^{e} \frac{dx}{x\ln x} = \int_{0}^{1} \frac{du}{u} = \ln 1 - \ln 0 = 0 - -\infty = +\infty$

3. $\int \frac{1}{x^2(x-1)}\,dx =$

a. $\ln|x-1| + \frac{1}{x} + C$
b. $\ln|x^2(x-1)| + C$
c. $\ln|x| - \frac{1}{x} - \ln|x-1| + C$
d. $-\ln|x| + \frac{1}{x} + \ln|x-1| + C$ Correct Choice
e. $\ln|x-1| - \frac{1}{x} + C$

Solution: $\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \qquad 1 = Ax(x-1) + B(x-1) + Cx^2$

$x = 0: \Rightarrow B = -1 \qquad x = 1: \Rightarrow C = 1 \qquad$ Coeff of x^2: $0 = A + C \Rightarrow A = -1$

$$\int \frac{1}{x^2(x-1)}\,dx = \int\left(-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\right)dx = -\ln|x| + \frac{1}{x} + \ln|x-1| + C$$

4. By substituting $x = 3\tan\theta$, the integral $\int_0^3 x^2\sqrt{x^2+9}\,dx$ becomes

a. $\int_0^{\pi/4} 27\tan^2\theta\sec\theta\,d\theta$

b. $\int_0^{3} 27\tan^2\theta\sec^3\theta\,d\theta$

c. $\int_0^{\pi/4} 81\tan^3\theta\sec^2\theta\,d\theta$

d. $\int_0^{\pi/4} 81\tan^2\theta\sec^2\theta\,d\theta$

e. $\int_0^{\pi/4} 81\tan^2\theta\sec^3\theta\,d\theta$ Correct Choice

Solution: $x^2 = 9\tan^2\theta$ $\sqrt{x^2+9} = \sqrt{9\tan^2\theta+9} = 3\sec\theta$ $dx = 3\sec^2\theta\,d\theta$

$\int_0^{\pi/4} 9\tan^2\theta \cdot 3\sec\theta \cdot 3\sec^2\theta\,d\theta = \int_0^{\pi/4} 81\tan^2\theta\sec^3\theta\,d\theta$

5. Find the length of the curve $x = t^2$, $y = t^3$, for $0 \le t \le 1$.

a. $\frac{1}{27}\left(13\sqrt{13} - 8\right)$ Correct Choice

b. $\frac{2\pi}{27}\left(13\sqrt{13} - 8\right)$

c. $\frac{1}{27}\left(13\sqrt{13} - 1\right)$

d. $\frac{1}{27}$

e. $\frac{2\pi}{27}\left(13\sqrt{13} - 1\right)$

Solution: $L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\,dt = \int_0^1 t\sqrt{4+9t^2}\,dt$

$u = 4 + 9t^2$ $L = \frac{1}{18}\int_4^{13}\sqrt{u}\,du = \left[\frac{1}{18}\frac{2}{3}u^{3/2}\right]_4^{13} = \frac{1}{27}\left(13\sqrt{13} - 8\right)$

6. Which of the following integrals gives the surface area obtained by rotating the curve $y = e^{-4x}$, for $0 \le x \le 1$, about the y-axis?

a. $\int_0^1 2\pi e^{-4x}\sqrt{1+16e^{-8x}}\,dx$

b. $\int_0^1 2\pi x\sqrt{1+16e^{-8x}}\,dx$ Correct Choice

c. $\int_1^{e^{-4}} 2\pi y\sqrt{1+\frac{1}{16y^2}}\,dy$

d. $\int_0^1 \frac{\pi}{2}\sqrt{16y^2+1}\,dy$

e. $\int_0^1 \frac{\pi}{8}\frac{\ln y}{y}\sqrt{16y^2+1}\,dy$

Solution: x-integral because $y = f(x)$ and $0 \le x \le 1$. $r = x$ because y-axis.

$A = \int_0^1 2\pi r\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 2\pi x\sqrt{1+(-4e^{-4x})^2}\,dx = \int_0^1 2\pi x\sqrt{1+16e^{-8x}}\,dx$

7. Which of the following statements is true regarding the improper integral $\int_1^\infty \frac{dx}{e^x + \sqrt{x}}$?

a. The integral converges because $\int_1^\infty \frac{dx}{e^x + \sqrt{x}} < \int_1^\infty \frac{dx}{\sqrt{x}}$ and $\int_1^\infty \frac{dx}{\sqrt{x}}$ converges.

b. The integral diverges because $\int_1^\infty \frac{dx}{e^x + \sqrt{x}} > \int_1^\infty \frac{dx}{\sqrt{x}}$ and $\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges.

c. The integral diverges because $\int_1^\infty \frac{dx}{e^x + \sqrt{x}} > \int_1^\infty \frac{dx}{e^x}$ and $\int_1^\infty \frac{dx}{e^x}$ diverges.

d. The integral converges because $\int_1^\infty \frac{dx}{e^x + \sqrt{x}} < \int_1^\infty \frac{dx}{e^x}$ and $\int_1^\infty \frac{dx}{e^x}$ converges.

Correct Choice

e. The integral converges to 0.

Solution: $\frac{1}{e^x + \sqrt{x}} < \frac{1}{\sqrt{x}}$ and $\frac{1}{e^x + \sqrt{x}} < \frac{1}{e^x}$ So (b) and (c) are wrong.

$\int_1^\infty \frac{dx}{\sqrt{x}}$ diverges and $\int_1^\infty \frac{dx}{e^x}$ converges So (a) and (c) are wrong.

$\frac{1}{e^x + \sqrt{x}} > 0$ So (e) is wrong. (d) is correct by the Comparison Test.

8. Find the integrating factor for the differential equation $(1 + x^2)y' = 2xy + 1 + x^2$.

a. $I(x) = 1 + x^2$

b. $I(x) = \frac{1}{1 + x^2}$ Correct Choice

c. $I(x) = 2(1 + x^2)$

d. $I(x) = \frac{2}{1 + x^2}$

e. The equation is not linear, so there is no integrating factor.

Solution: Standard form: $\frac{dy}{dx} - \frac{2x}{1 + x^2}y = 1$ $P(x) = -\frac{2x}{1 + x^2}$

$I(x) = e^{\int P(x)dx} = e^{-\ln\left(1+x^2\right)} = \frac{1}{1 + x^2}$.

9. If $y = f(x)$ is a solution of the Initial Value Problem: $\frac{dy}{dx} = x + y^2$ with $f(1) = 2$, then $f'(1) =$

a. 1

b. 2

c. 3

d. 4

e. 5 Correct Choice

$f'(x) = \frac{dy}{dx} = x + y^2$ $f'(2) = 1 + 2^2 = 5$

10. Solve the Initial Value Problem: $\dfrac{dy}{dx} = \dfrac{y+1}{x+1}$ with $y(0) = -2$. Then $y(1) =$

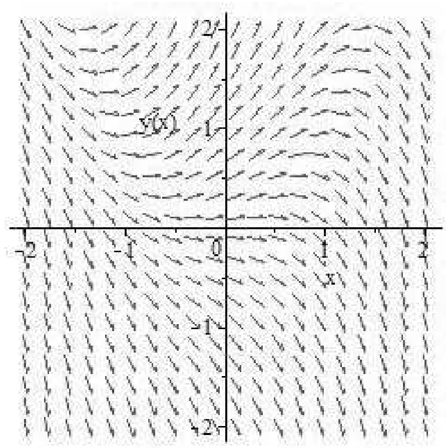
a. -1
b. -2
c. -3 Correct Choice
d. -4
e. -5

Solution: $\displaystyle\int \frac{dy}{y+1} = \int \frac{dx}{x+1}$ $\quad \ln|y+1| = \ln|x+1| + C \quad |y+1| = e^C|x+1|$

$y + 1 = \pm e^C(x+1) = A(x+1) \quad -2 + 1 = A(0+1) \quad A = -1 \quad y + 1 = -(x+1)$

$y = -x - 2 \quad y(1) = -3$

11. The plot at the right is the direction field for which differential equation?



a. $\dfrac{dy}{dx} = y - x^2$ Correct Choice

b. $\dfrac{dy}{dx} = y + x^2$

c. $\dfrac{dy}{dx} = x - y^2$

d. $\dfrac{dy}{dx} = x + y^2$

e. $\dfrac{dy}{dx} = y^2 - x$

Solution: The slope is zero when the right side is zero. Each right side is a parabola and the plot has zero slope on the parabola $y - x^2 = 0$.

PART II: **WORK OUT** (**10 pts each**)

Show all your work neatly and concisely and *box your final answer*. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

12. Find the surface area obtained by rotating the curve $y = \dfrac{x^2}{4} - \dfrac{1}{2}\ln x$, for $1 \le x \le 2$, about the y-axis.

Solution: x-integral because $y = f(x)$ and $1 \le x \le 2$. $\quad r = x$ because y-axis.

$$A = \int_1^2 2\pi r\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 2\pi x\sqrt{1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2}\,dx = \int_1^2 2\pi x\sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}\right)}$$

$$= \int_1^2 2\pi x\sqrt{\frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}}\,dx = \int_1^2 2\pi x\left(\frac{x}{2} + \frac{1}{2x}\right)dx = \pi\int_1^2 (x^2+1)\,dx = \pi\left[\frac{x^3}{3} + x\right]_1^2$$

$$= \pi\left(\frac{8}{3} + 2\right) - \pi\left(\frac{1}{3} + 1\right) = \frac{10}{3}\pi$$

13. Integrate $\int \sqrt{16-9x^2}\, dx$.

Solution: $3x = 4\sin\theta \qquad 3\,dx = 4\cos\theta\, d\theta \qquad \sqrt{16-9x^2} = \sqrt{16-16\sin^2\theta} = 4\cos\theta$

$$\int \sqrt{16-9x^2}\, dx = \int 4\cos\theta \frac{4}{3}\cos\theta\, d\theta = \frac{16}{3}\int \frac{1+\cos 2\theta}{2}\, d\theta = \frac{8}{3}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$

Draw a triangle or: $\sin\theta = \dfrac{3x}{4} \qquad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\dfrac{9x^2}{16}}$

$$\theta = \arcsin\frac{3x}{4} \qquad \frac{\sin 2\theta}{2} = \sin\theta\cos\theta = \frac{3x}{4}\sqrt{1-\frac{9x^2}{16}}$$

$$\int \sqrt{16-9x^2}\, dx = \frac{8}{3}\left(\arcsin\frac{3x}{4} + \frac{3x}{4}\sqrt{1-\frac{9x^2}{16}}\right) + C$$

14. Integrate $\int \dfrac{4x^2-1}{(x^2+1)(x-2)}\, dx$.

Solution: $\dfrac{4x^2-1}{(x^2+1)(x-2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-2} \qquad 4x^2-1 = (Ax+B)(x-2) + C(x^2+1)$

$$4x^2-1 = (Ax+B)(x-2) + C(x^2+1) = (A+C)x^2 + (B-2A)x + (C-2B)$$

$$A+C = 4 \qquad B-2A = 0 \qquad C-2B = -1$$

$$C = 4-A \qquad B = 2A \qquad 4-A-4A = -1 \qquad 5A = 5 \qquad A = 1 \qquad B = 2 \qquad C = 3$$

$$\int \frac{4x^2-1}{(x^2+1)(x-2)}\, dx = \int \frac{x+2}{x^2+1} + \frac{3}{x-2}\, dx = \int \frac{x}{x^2+1} + \frac{2}{x^2+1} + \frac{3}{x-2}\, dx$$

$$= \frac{1}{2}\ln(x^2+1) + 2\arctan x + 3\ln|x-2| + C$$

15. When a dead body was found by the police its temperature was 85°F. After 2 hours its temperature dropped to 75°F. How many hours before the body was found was its temperature 95° F, assuming the room temperature was a constant 65° F?

Solution:

$$\frac{d(T-T_0)}{dt} = -k(T-T_0) \qquad T - T_0 = Ae^{-kt} \qquad T_0 = 65, \quad T(0) = 85, \quad T(2) = 75$$

$$t = 0: \qquad 85 - 65 = Ae^0 \qquad A = 20$$

$$t = 2: \qquad 75 - 65 = 20e^{-2k} \qquad e^{-2k} = \frac{1}{2} \qquad -2k = \ln\left(\frac{1}{2}\right) \qquad k = \frac{1}{2}\ln(2)$$

$$95 - 65 = 20e^{-\ln(2)t/2} \qquad e^{-\ln(2)t/2} = \frac{3}{2} \qquad -\frac{\ln(2)t}{2} = \ln\left(\frac{3}{2}\right)$$

$$t = -2\frac{\ln(3/2)}{\ln(2)} \text{ hours}$$

16. A bathtub starts out with 10 gallons of water containing 5 pounds of bath salts. Pure water is entering the bathtub at 0.2 gallons per minute and bath salts are poured in at 3 pound per minute. The water is kept well mixed and drains out at 0.2 gallons per minute. How long will it take until there are 10 pounds of bath salts in the bathtub?

Solution: $$\frac{dS}{dt}\frac{\text{lb}}{\text{min}} = 3\frac{\text{lb}}{\text{min}} - \frac{0.2\text{ gal}}{\text{min}}\frac{S\text{ lb}}{10\text{ gal}} \qquad S(0) = 5$$

Standard Linear form: $$\frac{dS}{dt} + .02S = 3 \qquad P(t) = .02 \qquad I = e^{\int P(t)dt} = e^{.02t}$$

$$e^{.02t}\frac{dS}{dt} + .02e^{.02t}S = 3e^{.02t} \qquad \frac{d}{dt}\left(e^{.02t}S\right) = 3e^{.02t} \qquad e^{.02t}S = \int 3e^{.02t}dt = 3\frac{e^{.02t}}{.02} + C$$

$$e^{.02t}S = 150e^{.02t} + C$$

$S(0) = 5$:

$$e^0 5 = 150e^0 + C \qquad C = -145 \qquad e^{.02t}S = 150e^{.02t} - 145 \qquad S = 150 - 145e^{-.02t}$$

$$10 = 150 - 145e^{-.02t} \qquad 145e^{-.02t} = 140 \qquad -.02t = \ln\frac{140}{145} \qquad t = 50\ln\frac{29}{28}$$

17. A recursive sequence is defined by $a_1 = 2, \quad a_{n+1} = 5 - \dfrac{4}{a_n}$.

a. Use Mathematical Induction to prove:

The sequence is increasing and bounded between 0 and 5, i.e.

$P(n): \quad a_{n+1} > a_n \quad \text{and} \quad 0 < a_n < 5.$

i. Initialization Step: Find some terms and verify $P(1)$ and $P(2)$ are true:

Solution: $a_1 = 2, \quad a_2 = 5 - \dfrac{4}{2} = 3, \quad a_3 = 5 - \dfrac{4}{3} = \dfrac{11}{3},$

$P(1): \quad a_2 = 3 > a_1 = 2 \quad \text{and} \quad 0 < a_1 = 2 < 5.$ True

$P(2): \quad a_3 = \dfrac{11}{3} > a_2 = 3 \quad \text{and} \quad 0 < a_2 = 3 < 5.$ True

ii. Write out $P(k)$ and $P(k+1)$:

Solution:

$P(k): \quad a_{k+1} > a_k \quad \text{and} \quad 0 < a_k < 5.$

$P(k+1): \quad a_{k+2} > a_{k+1} \quad \text{and} \quad 0 < a_{k+1} < 5.$

iii. Induction Step: Assuming $P(k)$ prove $P(k+1)$:

Solution:

$a_{k+1} > a_k > 0$ implies $\dfrac{1}{a_{k+1}} < \dfrac{1}{a_k}$ implies $\dfrac{-4}{a_{k+1}} > \dfrac{-4}{a_k}$

implies $5 - \dfrac{4}{a_{k+1}} > 5 - \dfrac{4}{a_k}$ implies $a_{k+2} > a_{k+1}$

So the sequence is increasing.

$0 < a_k < 5$ implies $\dfrac{1}{a_k} > \dfrac{1}{5}$ implies $\dfrac{-4}{a_k} < \dfrac{-4}{5}$

implies $5 - \dfrac{4}{a_k} < 5 - \dfrac{4}{5}$ implies $a_{k+1} < 5$

Also since the sequence is increasing from $a_1 = 2$, we have $0 < a_{k+1}$.

So the sequence is bounded between 0 and 5.

b. What Theorem guarantees the sequence converges? Find the limit.

Solution: The Bounded Monotonic Sequence Theorem says the sequence converges.

Let $L = \lim\limits_{n\to\infty} a_n$. Then $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty}\left(5 - \dfrac{4}{a_n}\right) = 5 - \dfrac{4}{\lim\limits_{n\to\infty} a_n}$

So $L = 5 - \dfrac{4}{L} \qquad L^2 - 5L + 4 = 0 \qquad (L-4)(L-1) = 0$ Limit must be 1 or 4.

Since a_n is increasing from 2, the limit must be 4.