MATH 152 Honors FINAL EXAM Spring 2012

Sections 201-202 Solutions P. Yasskin

Multiple Choice: (15 problems, 4 points each)

**1.** Compute $\int_1^e 9x^2 \ln x\, dx$

**a.** $2e^3 + 1$ CORRECT

**b.** $2e^3 - 2$

**c.** $2e^3$

**d.** $3e^3 - 3e^2$

**e.** $3e^3 - 3e^2 + 3$

$$u = \ln x \qquad dv = 9x^2\, dx$$
$$du = \frac{1}{x}\, dx \qquad v = 3x^3$$
$$\int_1^e 3x^2 \ln x\, dx = 3x^3 \ln x - \int 3x^2\, dx = \left[3x^3 \ln x - x^3\right]_1^e = 2e^3 + 1$$

**2.** Compute $\int_1^2 \frac{1}{(x-2)^{4/3}}\, dx$

**a.** $-\infty$

**b.** $-3$

**c.** $-1$

**d.** $3$

**e.** $\infty$ CORRECT

$$\int_1^2 \frac{1}{(x-2)^{4/3}}\, dx = \left[-3(x-2)^{-1/3}\right]_1^2 = \lim_{x\to 2^-} \frac{-3}{(x-2)^{1/3}} + \frac{3}{(1-2)^{1/3}} = \infty$$

**3.** Find the arclength of the parametric curve $x = t^4 \quad y = \frac{1}{2}t^6$ for $0 \le t \le 1$.

**a.** $\frac{61}{54}$ CORRECT

**b.** $\frac{16}{9}$

**c.** $\frac{11}{9}$

**d.** $\frac{1}{9}$

**e.** $\frac{1}{54}$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{(4t^3)^2 + (3t^5)^2}\, dt = \int_0^1 \sqrt{16t^6 + 9t^{10}}\, dt = \int_0^1 t^3\sqrt{16 + 9t^4}\, dt$$
$$u = 16 + 9t^4 \qquad du = 36t^3\, dt \qquad \frac{1}{36}\, du = t^3\, dt$$
$$L = \frac{1}{36}\int_{16}^{25} \sqrt{u}\, du = \frac{1}{36}\left[\frac{2u^{3/2}}{3}\right]_{16}^{25} = \frac{1}{54}(25^{3/2} - 16^{3/2}) = \frac{1}{54}(125 - 64) = \frac{61}{54}$$

**4.** A $2$ meter bar has linear density $\rho = 1 + x^3$ kg/m where $x$ is measured from one end. Find the average density of the bar.

**a.** $2$ kg/m
**b.** $3$ kg/m CORRECT
**c.** $4.5$ kg/m
**d.** $5$ kg/m
**e.** $6$ kg/m

$$\rho_{\text{ave}} = \frac{1}{2}\int_0^2 (1+x^3)\,dx = \frac{1}{2}\left[x + \frac{x^4}{4}\right]_0^2 = \frac{1}{2}(2+4) = 3$$

**5.** A $2$ meter bar has linear density $\rho = 1 + x^3$ kg/m where $x$ is measured from one end. Find the center of mass of the bar.

**a.** $\frac{5}{7}$ m
**b.** $\frac{5}{6}$ m
**c.** $\frac{6}{5}$ m
**d.** $\frac{7}{5}$ m CORRECT
**e.** $\frac{42}{5}$ m

$$M = \int_0^2 (1+x^3)\,dx = \left[x + \frac{x^4}{4}\right]_0^2 = 2 + 4 = 6$$
$$M_1 = \int_0^2 x(1+x^3)\,dx = \left[\frac{x^2}{2} + \frac{x^5}{5}\right]_0^2 = \frac{42}{5}$$
$$\bar{x} = \frac{M_1}{M} = \frac{42}{5 \cdot 6} = \frac{7}{5}$$

**6.** If $y(x)$ satisfies the differential equation $\frac{dy}{dx} = \frac{x}{y}$ and the initial condition $y(0) = 3$, find $y(4)$.

**a.** $1$
**b.** $2$
**c.** $3$
**d.** $4$
**e.** $5$ CORRECT

$$\int y\,dy = \int x\,dx \quad \Rightarrow \quad \frac{y^2}{2} = \frac{x^2}{2} + C \quad \Rightarrow \quad y = \pm\sqrt{x^2 + 2C}$$
$$x = 0 \quad \& \quad y = 3 \quad \Rightarrow \quad \frac{9}{2} = \frac{0}{2} + C \quad \Rightarrow \quad C = \frac{9}{2} \quad \Rightarrow \quad y = \sqrt{x^2+9} \quad \Rightarrow \quad y(4) = 5$$

**7.** Find an integrating factor for the differential equation $\frac{dy}{dx} = 2xy + \sin x$.

**a.** $e^{-\cos x}$
**b.** $e^{-\sin x}$
**c.** $e^{\cos x}$
**d.** $e^{x^2}$
**e.** $e^{-x^2}$ CORRECT

$$\frac{dy}{dx} - 2xy = \sin x \quad \Rightarrow \quad P = -2x \quad \Rightarrow \quad I = e^{\int P\,dx} = e^{-x^2}$$

**8.** A sequence is defined recursively by: $a_1 = 4$ and $a_{n+1} = \sqrt{10a_n - 16}$. Find $\lim_{n\to\infty} a_n$.

**a.** 2
**b.** 4
**c.** 6
**d.** 8 CORRECT
**e.** Diverges

If $L = \lim_{n\to\infty} a_n$ exists, then $L = \sqrt{10L - 16}$.
So $L^2 - 10L + 16 = 0$ or $(L-2)(L-8) = 0$ or $L = 2$ or $8$.
$a_1 = 4$ $\quad a_2 = \sqrt{10 \cdot 4 - 16} = \sqrt{24} > 4 = a_1$.
So $a_n$ is increasing from 4 and the limit is probably 8.

To see this rigorously, let $P(n)$ be the statement $a_{n+1} > a_n$ and $4 \le a_n \le 8$.
Since $a_2 > a_1$ and $4 \le a_1 = 4 \le 8$, we have $P(1)$ is true.
Assume $P(k)$ is true, i.e. $a_{k+1} > a_k$ and $4 \le a_k \le 8$.
Then $10a_{k+1} > 10a_k$ and $40 \le 10a_k \le 80$ and $10a_{k+1} - 16 > 10a_k - 16$ and $24 \le 10a_k - 16 \le 64$
Since $10a_k - 16 \ge 24 > 0$, we can take square roots without modifying inequalities.
So $\sqrt{10a_{k+1} - 16} > \sqrt{10a_k - 16}$ and $\sqrt{24} \le \sqrt{10a_k - 16} \le \sqrt{64}$
Or $a_{k+2} > a_{k+1}$ and $4 \le \sqrt{24} \le a_{k+1} \le \sqrt{64} = 8$ which is $P(k+1)$.
So $a_n$ is increasing and bounded above by 8. So the limit exists and must be 8.

**9.** $\sum_{n=2}^{\infty} \frac{3^n}{2^{2n-1}} =$

**a.** 2
**b.** $\frac{9}{14}$
**c.** $\frac{9}{2}$ CORRECT
**d.** 4
**e.** Diverges

$$a = \frac{3^2}{2^{4-1}} = \frac{9}{8} \qquad r = \frac{3}{4} \qquad |r| < 1 \qquad \sum_{n=2}^{\infty} \frac{3^n}{2^{2n-1}} = \frac{\frac{9}{8}}{1 - \frac{3}{4}} = \frac{9}{8-6} = \frac{9}{2}$$

**10**. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{2^n}{(n+1)^2}(x-3)^n$.

**a**. $0$
**b**. $\frac{1}{3}$
**c**. $\frac{1}{2}$ CORRECT
**d**. $2$
**e**. $3$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-3)^{n+1}}{(n+2)^2}\frac{(n+1)^2}{2^n(x-3)^n}\right| = 2|x-3|$$

Convergent if $L = 2|x-3| < 1$ or $|x-3| < \frac{1}{2}$. So $R = \frac{1}{2}$

**11**. $\lim_{x\to0}\frac{\sin x - x\cos x}{x^3} =$

**a**. $\frac{1}{6}$
**b**. $\frac{1}{3}$ CORRECT
**c**. $\frac{1}{2}$
**d**. $\frac{2}{3}$
**e**. $\infty$

$\sin x = x - \frac{x^3}{3!} + \cdots = x - \frac{x^3}{6} + \cdots$ $\qquad \cos x = 1 - \frac{x^2}{2} + \cdots$ $\qquad x\cos x = x - \frac{x^3}{2} + \cdots$

$$\lim_{x\to0}\frac{\sin x - x\cos x}{x^3} = \lim_{x\to0}\frac{\left(x - \frac{x^3}{6} + \cdots\right) - \left(x - \frac{x^3}{2} + \cdots\right)}{x^3} = \lim_{x\to0}\frac{-\frac{x^3}{6} + \frac{x^3}{2} + \cdots}{x^3} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

**12**. Suppose the series $\sum_{n=1}^{\infty} n e^{(-n^2)}$ is approximated by its $9^{\text{th}}$ partial sum $\sum_{n=1}^{9} n e^{(-n^2)}$.
Use an integral to bound the error in this approximation.

**a**. $\frac{1}{2}e^{-64}$
**b**. $\frac{1}{2}e^{-81}$ CORRECT
**c**. $\frac{1}{2}e^{-100}$
**d**. $\frac{1}{2}e^{-121}$
**e**. $\frac{1}{2}e^{-144}$

The error is $E = \sum_{n=10}^{\infty} n e^{(-n^2)}$.

So $E \le \int_9^{\infty} n e^{(-n^2)}\,dn = -\frac{1}{2}e^{(-n^2)}\Big|_9^{\infty} = 0 - -\frac{1}{2}e^{(-9^2)} = \frac{1}{2}e^{-81}$

**13**. Find the angle between the vectors $\vec{u} = \langle 1, 1, -1 \rangle$ and $\vec{v} = \langle 1, -2, -1 \rangle$.

**a**. $0°$
**b**. $30°$
**c**. $45°$
**d**. $60°$
**e**. $90°$ CORRECT

$\vec{u} \bullet \vec{v} = 1 - 2 + 1 = 0 \qquad \cos\theta = \dfrac{\vec{u} \bullet \vec{v}}{|\vec{u}||\vec{v}|} = 0 \qquad \theta = 90°$

**14**. If $\vec{u}$ points South-West and $\vec{v}$ points Up, which way does $\vec{u} \times \vec{v}$ point?

**a**. South-East
**b**. North-East
**c**. North-West CORRECT
**d**. 45° Up from North-West
**e**. 45° Down from North-West

Hold your right fingers South-West with the palm facing Up. Then your thumb points North-West.

**15**. Find a unit vector perpendicular to both $\vec{a} = (3, -2, 1)$ and $\vec{b} = (-1, 0, 1)$.

**a**. $(-2, -4, -2)$
**b**. $(-2, 4, -2)$
**c**. $(1, -2, 1)$
**d**. $\left( \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$ CORRECT
**e**. $\left( \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \vec{i}(-2 - 0) - \vec{j}(3 + 1) + \vec{k}(0 - 2) = (-2, -4, -2)$$

$$\left|\vec{a} \times \vec{b}\right| = \sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$$

$$\frac{\vec{a} \times \vec{b}}{\left|\vec{a} \times \vec{b}\right|} = \frac{1}{2\sqrt{6}}(-2, -4, -2) = \left( \frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{-1}{\sqrt{6}} \right)$$

Work Out (4 questions, 12 points each)

Show all you work.

**16**. Compute $\int_2^4 \frac{8}{x^3\sqrt{x^2-4}}\,dx$

$x = 2\sec\theta \qquad dx = 2\sec\theta\tan\theta\,d\theta$

$x = 2$ @ $\sec\theta = 1$ or $\theta = 0$ $\qquad$ $x = 4$ @ $\sec\theta = 2$ or $\theta = \frac{\pi}{3}$

$$\int_2^4 \frac{8}{x^3\sqrt{x^2-4}}\,dx = \int_0^{\pi/3} \frac{8 \cdot 2\sec\theta\tan\theta\,d\theta}{8\sec^3\theta\sqrt{4\sec^2\theta-4}} = \int_0^{\pi/3} \frac{\sec\theta}{\sec^3\theta}\,d\theta = \int_0^{\pi/3} \cos^2\theta\,d\theta$$

$$= \int_0^{\pi/3} \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/3} = \frac{1}{2}\left[\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right] = \frac{\pi}{6} + \frac{\sqrt{3}}{8}$$

**17**. The curve $y = x^2$ is rotated about the $y$-axis to form a bowl. If the bowl contains $8\pi$ cm$^3$ of water, what is the height of the water in the bowl?

The silce at height $y$ is a circle. So its area is $A = \pi r^2$.

The radius is $r = x = \sqrt{y}$ So the area is $A = \pi y$

and the volume of the slice of thickness $dy$ is $dV = \pi y\,dy$.

So the volume up to height $h$ is

$$V = \int dV = \int_0^h \pi y\,dy = \pi\left[\frac{y^2}{2}\right]_0^h = \frac{\pi}{2}h^2$$

We equate the volume to $8\pi$ and solve for $h$:

$\frac{\pi}{2}h^2 = 8\pi \qquad \Rightarrow \qquad h^2 = 16 \qquad \Rightarrow \qquad h = 4$ cm

**18.** A leaking sandbag is lifted $20$ ft at $2$ ft/sec. The sandbag starts out weighing $50$ lb but is leaking sand at $3$ lb/sec. How much work is done to lift the sandbag?
HINT: What is the weight of the bag when it is $y$ ft above the ground?

In $t$ sec the bag is lifted $y = 2\ \frac{\text{ft}}{\text{sec}} \cdot t\ \text{sec} = 2t$ ft.

During this time it loses $\Delta F = 3$ lb/sec$\cdot t$ sec $= 3t$ lb of sand.

So $\Delta F = 3\frac{y}{2}$ and the weight is $F = 50 - \Delta F = 50 - 3\frac{y}{2}$. So the work done is

$$W = \int_0^{20} F dy = \int_0^{20} \left(50 - 3\frac{y}{2}\right) dy = \left[50y - \frac{3y^2}{4}\right]_0^{20} = 50 \cdot 20 - \frac{3 \cdot 20^2}{4} = 700 \text{ ft-lb}$$

**19.** Determine if the series $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$ converges absolutely, converges but not absolutely or diverges.
If it converges, find the sum. If it diverges, does it diverge to $+\infty$, $-\infty$ or neither?

The related absolute series is $\sum_{n=0}^{\infty} \frac{2^n}{n!}$. Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we conclude $\sum_{n=0}^{\infty} \frac{2^n}{n!} = e^2$.

Alternatively, apply the ratio test: $a_n = \frac{2^n}{n!}$ $\quad a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{(n+1)} = 0 < 1$$

$\Rightarrow$ Absolute series converges and original series converges absolutely.

Similarly, since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we conclude $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} = e^{-2}$.

$\Rightarrow$ Original series converges to $e^{-2}$.

Circle One: **[Converges Absolutely]** Converges Conditionally Diverges

Fill in the Blank: Converges to ____$e^{-2}$____

Or Circle One: Diverges to $+\infty$ $-\infty$ Neither