MATH 152, SPRING 2012
COMMON EXAM I - VERSION A SOLUTIONS

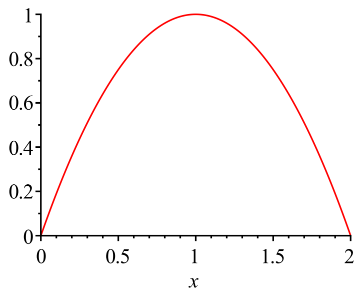
Last Name: ______________________________ **First Name:** ______________________________

Signature: ______________________________ **Section No:** ______________

PART I: Multiple Choice (4 pts each)

1. The region bounded by $y = 2x - x^2$ and $y = 0$ is revolved around the y-axis. Find the volume of the solid obtained.

   (a) $\frac{16\pi}{15}$
   (b) $\frac{8\pi}{3}$ CORRECT CHOICE
   (c) $\frac{8\pi}{15}$
   (d) $\frac{4\pi}{3}$
   (e) $\frac{4\pi}{15}$



Using the method of cylindrical shells, $V = \int_0^2 2\pi x(2x - x^2)\,dx = \frac{8\pi}{3}$.

2. $\int_0^{\pi/2} \sin^3 x \cos^4 x\,dx =$

   (a) $-\frac{2}{35}$
   (b) $\frac{1}{4}$
   (c) 0
   (d) $\frac{2}{35}$ CORRECT CHOICE
   (e) $\frac{1}{12}$

$\int_0^{\pi/2} \sin^3 x \cos^4 x\,dx = \int_0^{\pi/2} \sin^2 x \cos^4 x \sin x\,dx$

$= \int_0^{\pi/2} (1 - \cos^2 x)\cos^4 x \sin x\,dx$. Let $u = \cos x$. Then $du = -\sin x\,dx$.

Thus $\int_0^{\pi/2} (1 - \cos^2 x)\cos^4 x \sin x\,dx = -\int_1^0 (1 - u^2)u^4\,du = \int_0^1 (u^4 - u^6)\,du = \left(\frac{u^5}{5} - \frac{u^7}{7}\right)\Bigg|_0^1 = \frac{2}{35}$.

3. Find the average value of $f(x) = \dfrac{x}{\sqrt{x+1}}$ on the interval $[0, 3]$.

(a) $\dfrac{8}{9}$ CORRECT CHOICE

(b) $\dfrac{8}{3}$

(c) $\dfrac{14}{9}$

(d) $\dfrac{27}{32}$

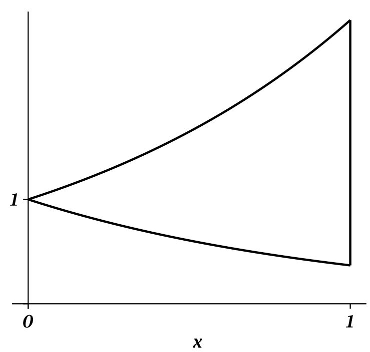
(e) $\dfrac{1}{288}$

$f_{ave} = \dfrac{1}{3}\displaystyle\int_0^3 \dfrac{x}{\sqrt{x+1}}\,dx$. Let $u = x+1$. Then $du = dx$.

Thus $\dfrac{1}{3}\displaystyle\int_0^3 \dfrac{x}{\sqrt{x+1}}\,dx = \dfrac{1}{3}\displaystyle\int_1^4 \dfrac{u-1}{\sqrt{u}}\,du = \dfrac{1}{3}\displaystyle\int_1^4 \left(u^{1/2} - u^{-1/2}\right)\,du$

$= \dfrac{1}{3}\left(\dfrac{2}{3}u^{3/2} - 2\sqrt{u}\right)\Big|_1^4 = \dfrac{8}{9}$.

4. Find the area bounded by $y = e^x$, $y = e^{-x}$, $x = 0$, $x = 1$.



(a) $e - \dfrac{1}{e}$

(b) $e - 2$

(c) $e + 2$

(d) $e + \dfrac{1}{e}$

(e) $-2 + e + \dfrac{1}{e}$ CORRECT CHOICE

$A = \displaystyle\int_0^1 (e^x - e^{-x})\,dx = \left(e^x + e^{-x}\right)\big|_0^1 = e + \dfrac{1}{e} - 2.$

5. $\displaystyle\int_1^e x \ln x \, dx =$

(a) $\frac{1}{4}e^2$

(b) $\frac{1}{4} - \frac{1}{4}e^2$

(c) $\frac{1}{4} + \frac{1}{4}e^2$ CORRECT CHOICE

(d) $-\frac{1}{4} + \frac{1}{2}e^2$

(e) $\frac{1}{6} + \frac{1}{2}e^2 - \frac{1}{6}e^3$

Integrate by parts, with $u = \ln x$ and $dv = x\,dx$. Then $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$.

$$\int_1^e x \ln x \, dx = \frac{x^2}{2} \ln x \bigg|_1^e - \int_1^e \frac{1}{x}\frac{x^2}{2}\,dx = \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\bigg|_1^e = \frac{e^2}{4} + \frac{1}{4}.$$

6. A spring has a natural length of 2 m. If a force of 12 N is required to hold the spring to a length of 4 m, find the work done to stretch the spring from 3 m to 6 m.

(a) 81 J

(b) 45 J CORRECT CHOICE

(c) 64 J

(d) 27 J

(e) 18 J

By Hooke's law, $f(2) = 12N$ where $f(x) = kx$. Thus $2k = 12$, so $k = 6$. The work needed to stretch the spring from 3 m to 6m is $W = \displaystyle\int_1^4 6x \, dx = 45$ Joules.

7. The region bounded by $y = x^2$ and $y = 4$ is revolved around the x-axis. Find the volume of the solid obtained.

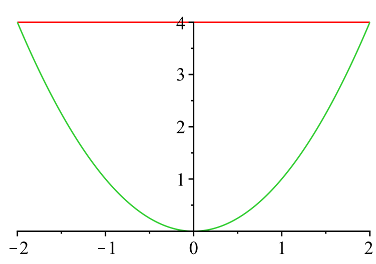
(a) $\frac{128\pi}{5}$

(b) $\frac{704\pi}{5}$

(c) $\frac{256\pi}{5}$ CORRECT CHOICE

(d) $\frac{88\pi}{3}$

(e) $\frac{128\pi}{3}$



Using the method of washers, $V = \displaystyle\int_{-2}^{2} \pi(16 - x^4)\,dx = \frac{256\pi}{5}$.

8. $\dfrac{d}{dx}\left[\displaystyle\int_0^{x^2} e^{-t^2}\,dt\right] =$

(a) $2xe^{-x^2}$
(b) e^{-x^4}
(c) $2xe^{-x^4}$ CORRECT CHOICE
(d) $2xe^{x^4}$
(e) e^{-x^2}

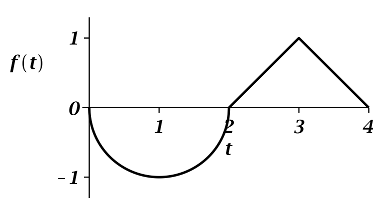
By the Fundamental Theorem of Calculus and the Chain Rule, $\dfrac{d}{dx}\left[\displaystyle\int_0^{x^2} e^{-t^2}\,dt\right] = 2xe^{-x^4}$.

9. $\displaystyle\int_0^{\pi/4} (\sec^2 x)e^{\tan x}\,dx =$

(a) $e^{\sqrt{2}/2} - 1$
(b) $e^{1/2} - 1$
(c) $e^{\sqrt{2}} - 1$
(d) $e - 1$ CORRECT CHOICE
(e) $1 - e$

Using integration by substitution, Let $u = \tan x$. Then $du = \sec^2 x\,dx$.
Thus $\displaystyle\int_0^{\pi/4} (\sec^2 x)e^{\tan x}\,dx = \int_0^1 e^u\,du = e^u\big|_0^1 = e - 1$.

10. Given the graph of $f(t)$ provided below, evaluate $\displaystyle\int_0^4 f(t)\,dt$. Note that the graph of $f(t)$ is a piece-wise defined function consisting of a semi-circle and lines.



(a) $1 - \dfrac{\pi}{2}$ CORRECT CHOICE
(b) $1 + \dfrac{\pi}{2}$
(c) $2 - \dfrac{\pi}{2}$
(d) $2 + \dfrac{\pi}{2}$
(e) $1 - \pi$

$\displaystyle\int_0^4 f(t)\,dt =$ -(Area enclosed by a semicircle of radius 1)+(Area enclosed by a triangle with base-length 2 and height 1) $= -\dfrac{1}{2}\pi(1)^2 + \dfrac{1}{2}(2)(1)$.

11. $\displaystyle\int \tan^6 x \sec^4 x\, dx =$

(a) $\frac{1}{9}\tan^9 x - \frac{1}{7}\tan^7 x + C$

(b) $\frac{1}{8}\tan^8 x - \frac{1}{6}\tan^6 x + C$

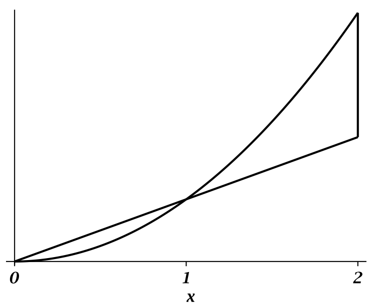
(c) $\left(\frac{1}{7}\tan^7 x\right)\left(\frac{1}{5}\sec^5 x\right) + C$

(d) $-\frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C$

(e) $\frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + C$ CORRECT CHOICE

$\displaystyle\int \tan^6 x \sec^4 x\, dx = \int \tan^6 x \sec^2 x \sec^2 x\, dx = \int \tan^6 x(\tan^2 x + 1)\sec^2 x\, dx$. Let $u = \tan x$. Then $du = \sec^2 x\, dx$. Thus $\displaystyle\int \tan^6 x(\tan^2 x + 1)\sec^2 x\, dx = \int u^6(u^2+1)\, du = \int (u^8 + u^6)\, du = \frac{u^9}{9} + \frac{u^7}{7} + C = \frac{\tan^9 x}{9} + \frac{\tan^7 x}{7} + C.$

12. Find the area bounded by $y = x$, $y = x^2$, $x = 0$ and $x = 2$.



(a) 2

(b) $\frac{1}{6}$

(c) $\frac{2}{3}$

(d) $\frac{5}{6}$

(e) 1 CORRECT CHOICE

The curves $y = x$, $y = x^2$ intersect at the point $(1, 1)$. The enclosed area equals

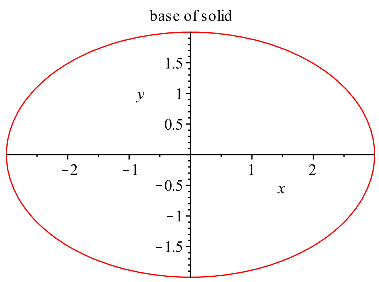
$A = \displaystyle\int_0^1 (x - x^2)\, dx + \int_1^2 (x^2 - x)\, dx = 1.$

PART II: WORK OUT (52 total points)

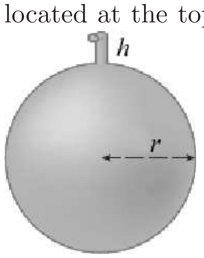
Directions: Present your solutions in the space provided. *Show all your work* neatly and concisely and *box your final answer*. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

13. (10 pts) Find the volume of the solid S described here: The base of S is the ellipse $4x^2 + 9y^2 = 36$. Cross sections perpendicular to the y-axis are squares.

A typical cross section of S consists of a square whose base is the horizontal line segment from the point $(-x, y)$ to (x, y), where $x \geq 0$, and the point (x, y) lies on the ellipse; that is, $4x^2 + 9y^2 = 36$. The length of the line segment is $2x$, so the area of a typical cross section is given by $(2x)^2 = 4x^2 = 36 - 9y^2$. Hence $V = \int_{-2}^{2} (36 - 9y^2)\, dy = 96$.



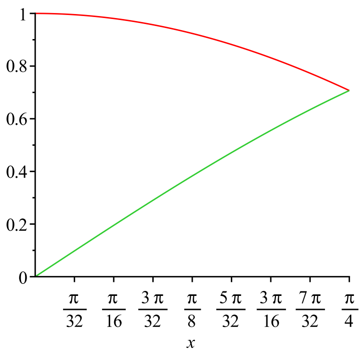
14. (10 pts) A tank is in the shape of a sphere of radius $r = 2$ m. The tank is *half full* of water with weight density $\rho g = 9800$ Newtons per cubic meter. Find the work done in pumping the water through a spout of length $h = 1$ m, located at the top of the tank. Suggestion: Place the axes so that the origin is located at the center of the tank.



The volume of an arbitrary horizontal 'slice' (layer) of water is given by $V = \pi x^2\, dy$. Since $x^2 + y^2 = 4$, $x^2 = 4 - y^2$. Thus $V = \pi(4 - y^2)\, dy$. In order to lift this slice of water to the top of the tank, we must exert a force of $F = \rho g \pi(4 - y^2)\, dy$ Newtons, where $\rho g = 9800 N/m^3$. This slice of water must be lifted $d = y + 3$ m to exit the tank (it has to pass through the top half of the sphere with radius 2 m and an extra 1 m to go through the spout). Hence the total work done to pump all the water out of the tank is $W = \int_0^2 \rho g \pi(4 - y^2)(y + 3)\, dy = 20\pi\rho g$ Joules.

15. Consider the region R bounded by $y = \cos x$, $y = \sin x$, $x = 0$, $x = \frac{\pi}{4}$.

a.) (4 pts) Sketch the region R.



b.) (5 pts) Set up the integral that gives the volume obtained by revolving the region R about the x axis using the method of washers. DO NOT EVALUATE THE INTEGRAL.

$$V = \int_0^{\pi/4} \pi \left(\cos^2 x - \sin^2 x\right)\, dx.$$

c.) (6 pts) Set up the integral that gives the volume obtained by revolving the region R about the line $x = \frac{\pi}{4}$ using the method of cylindrical shells. DO NOT EVALUATE THE INTEGRAL.

$$V = \int_0^{\pi/4} 2\pi \left(\frac{\pi}{4} - x\right)(\cos x - \sin x)\, dx.$$

16. Integrate:

a.) (8 pts) $\displaystyle\int \frac{\cos^2(\sqrt{x})}{\sqrt{x}}\, dx$

Let $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}\, dx$. Then $\displaystyle\int \frac{\cos^2(\sqrt{x})}{\sqrt{x}}\, dx = \int 2\cos^2 u\, du$

$$= 2\int \frac{1}{2}(1 + \cos(2u))\, du = \int (1 + \cos(2u))\, du = u + \frac{1}{2}\sin(2u) + C = \sqrt{x} + \frac{1}{2}\sin(2\sqrt{x}) + C.$$

b.) (9 pts) $\displaystyle\int_0^1 \arctan x\, dx$

Integrate by parts, with $u = \arctan(x)$ and $dv = dx$. Then $du = \frac{dx}{1 + x^2}$ and $v = x$. Hence

$\displaystyle\int_0^1 \arctan x\, dx = x \arctan x\Big|_0^1 - \int_0^1 \frac{x}{1 + x^2}\, dx = \left(x \arctan x - \frac{1}{2}\ln(1 + x^2)\right)\Bigg|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$, where the integral $\displaystyle\int \frac{x}{1 + x^2}\, dx$ is evaluated via the substitution $t = 1 + x^2$.

Last Name: ______________________________ **First Name**: ______________________________

Section No: _______________

Question	Points Awarded	Points
1-12		48
13		10
14		10
15		15
16		17
		100