MATH 152, SPRING 2012
COMMON EXAM III - VERSION A Solutions

Last Name: ______________________________ **First Name**: ______________________________

Signature: ______________________________ **Section No**: ______________

PART I: Multiple Choice (4 pts each)

1. What is the intersection of the sphere $(x-2)^2+(y-1)^2+(z-3)^2=16$ with the xy-plane?

   (a) The circle $(x-2)^2+(y-1)^2=16$.
   (b) The circle $(x-2)^2+(y-1)^2=1$.
   (c) The point $(2,1,0)$.
   (d) The circle $(x-2)^2+(y-1)^2=7$. Correct Answer
   (e) The sphere does not intersect the xy-plane.

   Solution: The sphere intersects the xy-plane when $z=0$. Substituting $z=0$ into the equation $(x-2)^2+(y-1)^2+(z-3)^2=16$ yields $(x-2)^2+(y-1)^2+(0-3)^2=16$, hence $(x-2)^2+(y-1)^2=7$.

2. If we apply the Ratio Test to the series $\sum_{n=1}^{\infty}\frac{(-4)^n n^2}{5^{n+1}}$, which of the following statements is true?

   (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=-\frac{4}{5}<1$, therefore the series converges.
   (b) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{4}{5}<1$, therefore the series converges. Correct Answer
   (c) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{4}{25}<1$, therefore the series converges.
   (d) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty>1$, therefore the series diverges.
   (e) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0<1$, therefore the series converges.

   Solution: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-4)^{n+1}(n+1)^2}{5^{n+2}}\frac{5^{n+1}}{(-4)^n(n^2)}\right|=\lim_{n\to\infty}\left|\frac{(-4)^n(-4)(n+1)^2}{5^{n+1}(5)}\frac{5^{n+1}}{(-4)^n(n^2)}\right|$
   $=\lim_{n\to\infty}\left|\frac{(-4)(n+1)^2}{5n^2}\right|=\frac{4}{5}<1$, therefore the series converges absolutely, therefore converges.

3. What does it mean to say $\sum_{n=1}^{\infty}a_n$ is convergent?

   (a) $\lim_{n\to\infty}a_n=0$.
   (b) $\lim_{n\to\infty}a_n$ exists.
   (c) There are numbers A and A' such that $A\le a_n\le A'$ for all n.
   (d) There are numbers B and B' such that $B\le a_1+a_2+...+a_n\le B'$ for all n.
   (e) $\lim_{n\to\infty}(a_1+a_2+...+a_n)$ exists. Correct Answer

   Solution: The partial sum is $s_n=a_1+a_2+...+a_n$ and $\sum_{n=1}^{\infty}a_n=\lim_{n\to\infty}s_n$.

4. Which of the following is a unit vector in the direction of $\mathbf{b} - 3\mathbf{a}$, where $\mathbf{a} = \langle 1, -2, 2 \rangle$ and $\mathbf{b} = \langle 5, 0, -3 \rangle$?

(a) $\left\langle -\frac{2}{11}, -\frac{6}{11}, \frac{9}{11} \right\rangle$

(b) $\left\langle \frac{2}{11}, \frac{6}{11}, -\frac{9}{11} \right\rangle$ Correct Answer

(c) $\langle 2, 6, -9 \rangle$

(d) $\left\langle -\frac{14}{\sqrt{524}}, -\frac{2}{\sqrt{524}}, \frac{18}{\sqrt{524}} \right\rangle$

(e) $\left\langle \frac{14}{\sqrt{524}}, \frac{2}{\sqrt{524}}, -\frac{18}{\sqrt{524}} \right\rangle$

Solution: $\mathbf{b} - 3\mathbf{a} = \langle 5, 0, -3 \rangle - 3\langle 1, -2, 2 \rangle = \langle 2, 6, -9 \rangle$. Divide by the magnitude to make this vector a unit vector. Since $|\langle 2, 6, -9 \rangle| = \sqrt{4 + 36 + 81} = 11$, $\mathbf{u} = \frac{\langle 2, 6, -9 \rangle}{11} = \left\langle \frac{2}{11}, \frac{6}{11}, -\frac{9}{11} \right\rangle$.

5. Represent $\frac{1}{1 + 4x^2}$ as a power series about 0 and give the interval of convergence.

(a) $\frac{1}{1 + 4x^2} = \sum_{n=0}^{\infty} (-4)^n x^{2n}$, where $|x| < \frac{1}{2}$. Correct Answer

(b) $\frac{1}{1 + 4x^2} = \sum_{n=0}^{\infty} (-4)^n x^{2n}$, where $|x| < \frac{1}{4}$.

(c) $\frac{1}{1 + 4x^2} = \sum_{n=0}^{\infty} (-4)^n x^{2n}$, where $|x| < 4$.

(d) $\frac{1}{1 + 4x^2} = \sum_{n=0}^{\infty} 4^n x^{2n}$, where $|x| < \frac{1}{2}$.

(e) $\frac{1}{1 + 4x^2} = \sum_{n=0}^{\infty} 4^n x^{2n}$, where $|x| < \frac{1}{4}$.

Solution: $\frac{1}{1 + 4x^2} = \sum_{n=0}^{\infty} (-4x^2)^n = \sum_{n=0}^{\infty} (-4)^n x^{2n}$, where $|-4x^2| < 1$, hence $|x| < \frac{1}{2}$.

6. Find the cosine of the angle, θ, between the vectors $\mathbf{a} = \langle 2, 0, 1 \rangle$ and $\mathbf{b} = \langle 1, -2, 1 \rangle$.

(a) $\cos(\theta) = \frac{4}{\sqrt{30}}$

(b) $\cos(\theta) = \frac{5}{\sqrt{30}}$

(c) $\cos(\theta) = \frac{3}{\sqrt{12}}$

(d) $\cos(\theta) = \frac{5}{\sqrt{12}}$

(e) $\cos(\theta) = \frac{3}{\sqrt{30}}$ Correct Answer

Solution: $\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\langle 2, 0, 1 \rangle \cdot \langle 1, -2, 1 \rangle}{|\langle 2, 0, 1 \rangle||\langle 1, -2, 1 \rangle|} = \frac{3}{\sqrt{30}}$

7. Find the 15th derivative at $x = 0$ for $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{7^n(n+5)} x^n$.

(a) $f^{(15)}(0) = 0$

(b) $f^{(15)}(0) = \frac{15!}{7^{15}(20)}$

(c) $f^{(15)}(0) = -\frac{15!}{7^{15}(20)}$ Correct Answer

(d) $f^{(15)}(0) = -\frac{15}{7^{15}(20)}$

(e) $f^{(15)}(0) = \frac{15}{7^{15}(20)}$

Solution: $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{7^n(n+5)} x^n = \frac{1}{5} - \frac{1}{7(6)}x + \frac{1}{7^2(7)}x^2 - \frac{1}{7^3(8)}x^3 + ... - \frac{1}{7^{15}(20)}x^{15} + \frac{1}{7^{16}(21)}x^{16} + ...$ Now, if the degree on x is less than 15, the 15th derivative of these terms will be zero. If the degree on x is greater than 15, the 15th derivative of these terms will still contain a power of x, and hence will equal 0 when $x = 0$. If the degree on x is 15, then $f^{(15)}(x^{15}) = 15!$ Thus $f^{15}(0) = -\frac{15!}{7^{15}(20)}$

8. Find the Taylor Polynomial of degree 2, $T_2(x)$, for $f(x) = \sqrt{x}$ centered at $a = 9$.

(a) $T_2(x) = -\frac{1}{216}(x-9)^2$

(b) $T_2(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2$ Correct Answer

(c) $T_2(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{108}(x-9)^2$

(d) $T_2(x) = -\frac{1}{108}(x-9)^2$

(e) None of these.

Solution: $T_2(x) = f(9) + f'(9)(x-9) + \frac{f''(9)}{2}(x-9)^2$.

$T_2(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2$.

9. The series $\sum_{n=1}^{\infty} ne^{-n^2}$

(a) converges by the Integral Test. Correct Answer

(b) diverges by the Ratio Test.

(c) diverges by the Test for Divergence.

(d) diverges by the p-series test.

(e) converges by the Comparison Test with $\sum_{n=1}^{\infty} e^{-n^2}$.

Solution: Since xe^{-x^2} is a continuous, positive, decreasing function on $[1, \infty)$, we may apply the integral test: $\int_1^{\infty} xe^{-x^2}\, dx = -\frac{1}{2}e^{-x^2}\Big|_1^{\infty} = \frac{1}{2e}$. Since the improper integral converges, so does the series.

10. In three-dimensional space, R^3, the equation $x^2 + y^2 = 9$ describes

   (a) a parabola.
   (b) a sphere.
   (c) a cylinder. Correct Answer
   (d) a circle.
   (e) a plane.

Solution: In two-dimension, $x^2 + y^2 = 9$ describes a circle. Thus, in three-dimension, as z varies, $x^2 + y^2 = 9$ describes a cylinder.

11. The series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

   (a) is absolutely convergent.
   (b) is convergent but not absolutely convergent. Correct Answer
   (c) is divergent.
   (d) is absolutely divergent but not divergent.
   (e) converges absolutely by the ratio test.

Solution: The series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges by the Alternating Series Test since the sequence $\left\{\frac{n}{n^2+1}\right\}$ decreases to zero. To test for absolute convergence, test the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ for convergence. Using the Limit Comparison Test, with $b_n = \frac{1}{n}$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = 1 > 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. Hence $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is converget but not absolutely convergent. Another solution: Since $\frac{x}{x^2+1}$ is continuous, positive and decreasing on $[1, \infty)$, we may use the integral test to show $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges:

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2} \ln(x^2+1)\Big|_1^{\infty} = \infty.$$

12. The series $\sum_{n=1}^{\infty} \frac{1}{e^n + \sqrt{n}}$ is

   (a) divergent because $\frac{1}{e^n + \sqrt{n}} > \frac{1}{e^n}$ and $\sum_{n=1}^{\infty} \frac{1}{e^n}$ is a divergent geometric series.
   (b) convergent because $\frac{1}{e^n + \sqrt{n}} < \frac{1}{e^n}$ and $\sum_{n=1}^{\infty} \frac{1}{e^n}$ is a convergent geometric series. Correct Answer
   (c) convergent because $\frac{1}{e^n + \sqrt{n}} < \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a convergent p series.
   (d) divergent because $\frac{1}{e^n + \sqrt{n}} < \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p series.
   (e) divergent because $\frac{1}{e^n + \sqrt{n}} > \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p series.

Solution: Since $\sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ is a convergent geometric series ($r = \frac{1}{e}$), and $\frac{1}{e^n + \sqrt{n}} < \frac{1}{e^n}$, it follows that $\sum_{n=1}^{\infty} \frac{1}{e^n + \sqrt{n}} < \sum_{n=1}^{\infty} \frac{1}{e^n}$, thus by the comparison test, since the larger series converges, so does the smaller series.

13. Find the sum of the series $\sum\limits_{n=0}^{\infty} \frac{(-2)^n}{n!}$.

(a) e^2
(b) $\cos(-2)$
(c) $\sin(-2)$
(d) 0
(e) e^{-2} Correct Answer

Solution: Since $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$, it follows that $e^{-2} = \sum\limits_{n=0}^{\infty} \frac{(-2)^n}{n!}$.

PART II: WORK OUT (48 points total)

<u>**Directions**</u>: Present your solutions in the space provided. *Show all your work* neatly and concisely and *box your final answer*. You will be graded not merely on the final answer, but also on the quality and correctness of the work leading up to it.

14. a.) (5 pts) Using the known Maclaurin series for $\sin x$, find the Maclaurin series for $\sin(x^3)$.

Solution: $\sin x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, thus $\sin(x^3) = \sum\limits_{n=0}^{\infty} \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$.

b.) (5 pts) Using the Maclaurin series for $\sin(x^3)$, compute $\lim\limits_{x\to 0} \frac{\sin(x^3) - x^3}{x^9}$.

Solution: $\lim\limits_{x\to 0} \frac{\sin(x^3)-x^3}{x^9} = \lim\limits_{x\to 0} \frac{\frac{(-1)^n x^{6n+3}}{(2n+1)!} - x^3}{x^9} = \lim\limits_{x\to 0} \frac{x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + ... - x^3}{x^9}$

$= \lim\limits_{x\to 0} \frac{x^9\left(-\frac{1}{3!} + \frac{x^6}{5!} - \frac{x^{12}}{7!} + ...\right)}{x^9} = -\frac{1}{3!} = -\frac{1}{6}$

15. (5 pts) Using the known Maclaurin series for e^x, find a Maclaurin series for e^{-x^2}.

Solution: $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$, thus $e^{-x^2} = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$.

b.) (5 pts) Expand the Maclaurin series for e^{-x^2} found above to degree 4. Use this 4^{th} degree Maclaurin polynomial to estimate $\int_0^1 e^{-x^2}\,dx$. Do not simplify.

Solution: $e^{-x^2} = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \approx 1 - x^2 + \frac{x^4}{2}$.

Thus $\int_0^1 e^{-x^2}\,dx \approx \int_0^1 \left(1 - x^2 + \frac{x^4}{2}\right) dx = \left(x - \frac{x^3}{3} + \frac{x^5}{10}\right)\Big|_0^1 = 1 - \frac{1}{3} + \frac{1}{10} = \frac{23}{30}$.

16. (10 pts) Find a power series about 0 for $f(x) = \ln(6x+5)$. What is the associated radius of convergence?

Solution: $\frac{d}{dx}(\ln(6x+5)) = \frac{6}{6x+5} = \frac{6}{5\left(1+\frac{6x}{5}\right)} = \frac{6}{5}\sum\limits_{n=0}^{\infty}\left(-\frac{6x}{5}\right)^n = \sum\limits_{n=0}^{\infty}(-1)^n\left(\frac{6}{5}\right)^{n+1} x^n$, where $\left|\frac{6x}{5}\right| < 1$.

Thus, $\ln(6x+5) = \int \sum\limits_{n=0}^{\infty}(-1)^n\left(\frac{6}{5}\right)^{n+1} x^n\,dx = C + \sum\limits_{n=0}^{\infty}(-1)^n\left(\frac{6}{5}\right)^{n+1}\frac{x^{n+1}}{n+1}$, where $|x| < \frac{5}{6}$. Choosing $x = 0$ to solve for C yields $C = \ln 5$. Thus $\ln(6x+5) = \ln 5 + \sum\limits_{n=0}^{\infty}(-1)^n\left(\frac{6}{5}\right)^{n+1}\frac{x^{n+1}}{n+1}$, $R = \frac{5}{6}$.

17. (10 pts) Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n4^n}$. Be sure to test the endpoints for convergence.

Solution: Using the Ratio Test, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(2x-1)^{n+1}}{(n+1)4^{n+1}} \frac{n4^n}{(2x-1)^n}\right| = \lim_{n\to\infty} \left|\frac{(2x-1)(n)}{(n+1)4}\right| = \left|\frac{2x-1}{4}\right|$.

The Ratio Test states if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges. Hence $\left|\frac{2x-1}{4}\right| < 1$, so $|2x-1| < 4$, thus $\left|x - \frac{1}{2}\right| < 2$. Solve this inequality for x: $-\frac{3}{2} < x < \frac{5}{2}$.

Test the endpoints of the interval for convergence :

Test $x = -\frac{3}{2}$: $\sum_{n=1}^{\infty} \frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the Alternating Series Test since the sequence $\left\{\frac{1}{n}\right\}$ decreases to zero.

Test $x = \frac{5}{2}$: $\sum_{n=1}^{\infty} \frac{(4)^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which diverges by the p-series test. Thus the interval of convergence is $-\frac{3}{2} \leq x < \frac{5}{2}$.

18. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2+1}$.

a.) (4 pts) Prove the series is absolutely convergent.

Solution: $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{2n^2+1}\right| = \sum_{n=1}^{\infty} \frac{1}{2n^2+1} < \sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series, it follows that $\sum_{n=1}^{\infty} \frac{1}{2n^2}$ converges, thus $\sum_{n=1}^{\infty} \frac{1}{2n^2+1}$ also converges by the comparison test. Hence $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2+1}$ is absolutely convergent.

b.) (2 pts) Find s_2, the sum of the first 2 terms, to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2+1}$.

Solution: $S_2 = \sum_{i=1}^{2} \frac{(-1)^i}{2i^2+1} = -\frac{1}{3} + \frac{1}{9} = -\frac{2}{9}$

c.) (2 pts) Find an upper bound on $|R_2|$, the absolute value of the remainder, in using s_2 to approximate $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2+1}$.

Solution: $|R_2| \leq |a_3|$, where $a_n = \frac{(-1)^n}{2n^2+1}$. Thus $|R_2| \leq \frac{1}{19}$.