MATH 152 FINAL EXAM Spring 2005

Sections 513,514 Version B Solutions P. Yasskin

Multiple Choice: (5 points each)

1. Find the area between the curves $y = x^2$ and $y = 2x + 3$.

a. $\frac{4}{3}$

b. $\frac{8}{3}$

c. $\frac{16}{3}$

d. $\frac{32}{3}$ CORRECT

e. $\frac{88}{3}$

We find where the curves intersect:

$x^2 = 2x + 3 \qquad x^2 - 2x - 3 = 0 \qquad (x+1)(x-3) = 0 \qquad x = -1, 3$

$$A = \int_{-1}^{3} (2x + 3 - x^2)\,dx = \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3} = (9 + 9 - 9) - \left(1 - 3 + \frac{1}{3}\right) = \frac{32}{3}$$

2. The base of a solid is the circle $x^2 + y^2 = 9$ and the cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

a. 9

b. 18

c. 36

d. 72

e. 144 CORRECT

x integral The side of a square is $s = 2y$. So the area of a square is $A(x) = 4y^2 = 4(9 - x^2)$

$$V = \int_{-3}^{3} A(x)\,dx = \int_{-3}^{3} 4(9 - x^2)\,dx = 4\left[9x - \frac{x^3}{3}\right]_{-3}^{3} = 8(27 - 9) = 144$$

3. Using a trigonometric substitution, the integral $\int \frac{dx}{x^2\sqrt{16 + x^2}}$ becomes

a. $\int \frac{\cos^3\theta}{16\sin^2\theta}\,d\theta$

b. $\int \frac{\cos^3\theta}{64\sin^2\theta}\,d\theta$

c. $\int \frac{\cos\theta}{16\sin^2\theta}\,d\theta$ CORRECT

d. $\int \frac{1}{64\sin^2\theta\cos\theta}\,d\theta$

e. $\int \frac{1}{16\sin^2\theta\cos\theta}\,d\theta$

$x = 4\tan\theta \qquad dx = 4\sec^2\theta\,d\theta$

$$I = \int \frac{4\sec^2\theta\,d\theta}{16\tan^2\theta\sqrt{16 + 16\tan^2\theta}} = \int \frac{\sec\theta\,d\theta}{16\tan^2\theta} = \int \frac{\cos\theta\,d\theta}{16\sin^2\theta}$$

4. Compute $\int \frac{2}{x(x-2)}\,dx.$

a. $\ln|x-2| - \ln|x| + C$ CORRECT
b. $\ln|x| - \ln|x-2| + C$
c. $\ln|x| + 2\ln|x-2| + C$
d. $\ln|x| - 2\ln|x-2| + C$
e. $2\ln|x-2| - \ln|x| + C$

Partial Fractions: $\frac{2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$ Clear denominator: $2 = A(x-2) + Bx$

$x = 0$: $2 = A(-2)$ $A = -1$ $x = 2$: $2 = B(2)$ $B = 1$

$\int \frac{2}{x(x-2)}\,dx = \int \frac{-1}{x} + \frac{1}{x-2}\,dx = \ln|x-2| - \ln|x| + C$

5. Use the Middle Sum Rule with $n = 4$ intervals to approximate the integral $\int_1^9 (9 + x^2)\,dx.$

a. 240
b. 312 CORRECT
c. $314\frac{1}{3}$
d. 320
e. 400

$\Delta x = \frac{9-1}{4} = 2$

$M_4 = \Delta x(f(2) + f(4) + f(6) + f(8)) = 2(13 + 25 + 45 + 73) = 312$

6. Solve the differential equation $\frac{dy}{dx} = \frac{4}{3}\frac{x^3}{y^2}$ with the initial condition $y(0) = 2.$

a. $y = 2x^{4/3}$
b. $y = x^{3/4} + 2$
c. $y = x^{4/3} + 2$
d. $y = \sqrt[4]{x^3 + 8}$
e. $y = \sqrt[3]{x^4 + 8}$ CORRECT

Separate: $\int 3y^2\,dy = \int 4x^3 dx$ Integrate: $y^3 = x^4 + C$

Use the initial condition: $2^3 = 0^4 + C$ $C = 8$

Substitute back: $y^3 = x^4 + 8$ Solve: $y = \sqrt[3]{x^4 + 8}$

7. A sequence $\{a_n\}$ is defined by $a_1 = 4$ and $a_{n+1} = \sqrt{5a_n{}^2 - 16}$. Find $\lim_{n\to\infty} a_n$.

a. -2

b. 2

c. $\sqrt{5}$

d. 4

e. Divergent CORRECT

If the limit exists,
then $L = \lim_{n\to\infty} a_n$ satisfies $L = \sqrt{5L^2 - 16}$, or $L^2 = 5L^2 - 16$, or $4L^2 = 16$ or $L = \pm 2$.

To find out which one or neither, we try some terms:

$a_1 = 4,\ a_2 = \sqrt{5 \cdot 16 - 16} = 8,\ a_3 = \sqrt{5 \cdot 64 - 16} = \sqrt{304}$

The terms are increasing because
if $a_{n+1} > a_n$, then $5a_{n+1}{}^2 - 16 > 5a_n{}^2 - 16$ and $a_{n+2} > a_{n+1}$.

So $\lim_{n\to\infty} a_n$ diverges to ∞.

8. Compute $\displaystyle\sum_{n=1}^{\infty} \frac{2^{3n+1}}{3^{2n+1}}$

a. $\frac{2}{3}$

b. $\frac{8}{9}$

c. $\frac{16}{3}$ CORRECT

d. $\frac{16}{9}$

e. Divergent

Geometric series: Ratio: $r = \frac{2^3}{3^2} = \frac{8}{9} < 1$ First term: $a = \frac{2^4}{3^3} = \frac{16}{27}$

$$\sum_{n=1}^{\infty} \frac{2^{3n+1}}{3^{2n+1}} = \frac{\frac{16}{27}}{1 - \frac{8}{9}} = \frac{16}{27} \cdot 9 = \frac{16}{3}$$

9. Compute $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!}$

HINT: Think about the standard Maclaurin series.

a. -1 CORRECT

b. 1

c. 2π

d. e^{π}

e. Divergent

Since $\cos x = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, we conclude $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n)!} = \cos\pi = -1$

10. A triangle has vertices $A = \left(2+\sqrt{2}, 3, 3\right)$, $B = (2,-1,-1)$ and $C = (2,2,2)$. Find the angle at vertex C.

- **a.** $\frac{\pi}{4}$
- **b.** $\frac{\pi}{3}$
- **c.** $\frac{\pi}{2}$
- **d.** $\frac{3\pi}{4}$ CORRECT
- **e.** $\frac{2\pi}{3}$

$$\overrightarrow{CA} = A - C = \left(\sqrt{2}, 1, 1\right) \qquad \overrightarrow{CB} = B - C = (0,-3,-3) \qquad \overrightarrow{CA} \cdot \overrightarrow{CB} = -6$$

$$\left|\overrightarrow{CA}\right| = \sqrt{2+1+1} = 2 \qquad \left|\overrightarrow{CB}\right| = \sqrt{9+9} = 3\sqrt{2}$$

$$\cos\theta = \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\left|\overrightarrow{CA}\right|\left|\overrightarrow{CB}\right|} = \frac{-6}{2 \cdot 3\sqrt{2}} = \frac{-1}{\sqrt{2}} \qquad \Rightarrow \qquad \theta = 135^\circ = \frac{3\pi}{4}$$

11. If $\vec{u}$ points Up and $\vec{v}$ points North-West, which way does $\vec{u} \times \vec{v}$ point?

- **a.** South-West CORRECT
- **b.** South-East
- **c.** North-East
- **d.** 45° Up from North-West
- **e.** 45° Down from North-West

Hold your right fingers Up with the palm facing North-West. Then your thumb points South-West.

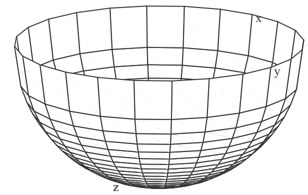
12. Find the area of a parallelogram with edges $\vec{a} = (-2,4,-1)$ and $\vec{b} = (3,0,2)$.

- **a.** 8
- **b.** $\sqrt{209}$ CORRECT
- **c.** 209
- **d.** $2\sqrt{2}$
- **e.** $\frac{1}{2}\sqrt{209}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 4 & -1 \\ 3 & 0 & 2 \end{vmatrix} = \vec{i}(8) - \vec{j}(-4+3) + \vec{k}(0-12) = (8,1,-12)$$

$$A = \left|\vec{a} \times \vec{b}\right| = \sqrt{64+1+144} = \sqrt{209}$$

13. (12 points) A water tank has the shape of a hemisphere with radius 5 m. It is filled with water to a height of 3 m. Find the work in Joules required to empty the tank by pumping all of the water to the top of the tank. Give your answer in terms of ρ (the density of water) and g (the acceleration of gravity).



Set $y = 0$ at the top of the tank and measure y downward.

The slice y below the top has volume $dV = \pi r^2\,dy = \pi(25 - y^2)\,dy$.

There is water from $y = 2$ to $y = 5$ below the top. So the work is

$$W = \int_2^5 \rho g y \pi (25 - y^2)\,dy = \rho g \pi \int_2^5 (25y - y^3)\,dy = \rho g \pi \left[\frac{25y^2}{2} - \frac{y^4}{4} \right]_2^5$$

$$= \rho g \pi \left(\frac{625}{2} - \frac{625}{4} \right) - \rho g \pi \left(\frac{25 \cdot 4}{2} - \frac{16}{4} \right) = \frac{441}{4} \rho g \pi$$

14. (12 points) Compute $\int_0^{\pi/4} \sec^3\theta \tan^3\theta\,d\theta.$

$u = \sec\theta \qquad du = \sec\theta\tan\theta\,d\theta \qquad \tan^2\theta = \sec^2\theta - 1 = u^2 - 1$

$$\int_0^{\pi/4} \sec^3\theta\tan^3\theta\,d\theta = \int_0^{\pi/4} \sec^2\theta\tan^2\theta\,\sec\theta\tan\theta\,d\theta = \int_1^{\sqrt{2}} u^2(u^2 - 1)\,du$$

$$= \left[\frac{u^5}{5} - \frac{u^3}{3} \right]_1^{\sqrt{2}} = \left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} \right) - \left(\frac{1}{5} - \frac{1}{3} \right) = \frac{2\sqrt{2}}{15} + \frac{2}{15}$$

15. $\big($12 points$\big)$ The curve $y = \frac{x^2}{4} - \frac{\ln x}{2}$ between $x = 1$ and $x = 2$ is rotated about the y-axis. Find the area of the resulting surface.

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 = 1 + \frac{1}{4}x^2 - \frac{1}{2} + \frac{1}{4x^2} = \frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$$

$$A = \int_1^2 2\pi x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi\int_1^2 x\left(\frac{x}{2} + \frac{1}{2x}\right) dx = \pi\int_1^2 (x^2 + 1)\, dx = \pi\left[\frac{x^3}{3} + x\right]_1^2$$

$$= \pi\left(\frac{8}{3} + 2\right) - \pi\left(\frac{1}{3} + 1\right) = \pi\left(\frac{7}{3} + 1\right) = \frac{10}{3}\pi$$

16. $\big($12 points$\big)$ Find the Maclaurin series (using $\sum$ notation) for $f(x) = \frac{2x}{(1-2x)^2}$ by manipulating the derivative of the series for $g(x) = \frac{1}{1-2x}$. What is the interval of convergence for $f(x)$ (including endpoints)? Justify your answers.

$g(x) = \frac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty} 2^n x^n$ which converges for $|2x| < 1$ or $|x| < \frac{1}{2}$.

Thus the center is $x = 0$ and the radius is $R = \frac{1}{2}$.

Then $g'(x) = \frac{2}{(1-2x)^2} = \sum_{n=0}^{\infty} 2^n n x^{n-1}$ which also has center is $x = 0$ and radius is $R = \frac{1}{2}$, since the derivative of a power series has the same radius of convergence.

So $f(x) = \frac{2x}{(1-2x)^2} = xg'(x) = \sum_{n=0}^{\infty} 2^n n x^n$.

The interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$ except we need to check the endpoints.

At $x = -\frac{1}{2}$: $f\left(-\frac{1}{2}\right) = \sum_{n=0}^{\infty} 2^n n\left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^{n-1} n$ which diverges by the n^{th} Term Divergence Test.

At $x = \frac{1}{2}$: $f\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} 2^n n\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} n$ which diverges by the n^{th} Term Divergence Test.

So the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.