Spring 2005 Math 152
Exam 2A: Solutions
Mon, 28/Mar ©2005, Art Belmonte

For specificity, lengths are in centimeters unless stated otherwise.

1. (d)

   - *For non-STEPS folks:* We have
   $$\bar{x} = \frac{\sum_{k=1}^{3} m_k x_k}{\sum_{k=1}^{3} m_k} = \frac{(3)(1)+(5)(-2)+(7)(3)}{3+5+7} = \frac{14}{15}.$$
   (Note that this only gives you the x-coordinate of the center of mass, not the entire center of mass.)
   - *For STEPS folks:* Let $\mathbf{p} = [3, 5, 7]$ be the row vector of masses and
   $$\mathbf{r} = \begin{bmatrix} 1 & 2 \\ -2 & 5 \\ 3 & 1 \end{bmatrix}$$
   be a matrix whose rows are position vectors of the points. Then $m = \text{sum}(\mathbf{p}) = 3+5+7 = 15$ is the total mass.
   The center of mass is
   $$\begin{aligned} [\bar{x}, \bar{y}] &= \frac{1}{m}(\mathbf{pr}) \\ &= \frac{1}{15}[3-10+21, 6+25+7] \\ &= \left[\frac{14}{15}, \frac{38}{15}\right]. \end{aligned}$$
   So the x-coordinate of the center of mass is $\bar{x} = \frac{14}{15}$. (*Remark:* **pr** represents matrix multiplication, realized by taking the dot products of rows with columns. This immediately extends to 3-D center of mass problems in Calc 3.)

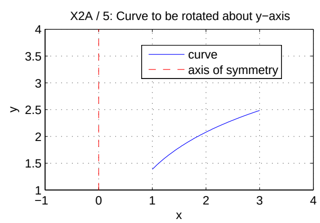
2. (a) Put the differential equation in standard linear form.
$$\frac{dy}{dx} + \frac{4x}{x^2+1} y = \frac{x^3}{x^2+1}$$
Then an integrating factor is
$$\begin{aligned} \mu = \exp\left(\int \frac{4x}{x^2+1}\,dx\right) &= \exp\left(2\ln\left(x^2+1\right)\right) \\ &= \exp\left(\ln\left(x^2+1\right)^2\right) \\ &= \left(x^2+1\right)^2. \end{aligned}$$

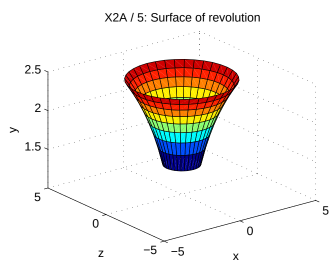
3. (e) The arc length of the curve
$$\mathbf{r}(t) = [x(t), y(t)] = [3t+1, 4-t], 1 \le t \le 3,$$
is $L = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$
$= \int_1^3 \sqrt{(3)^2 + (1)^2}\,dt = \int_1^3 \sqrt{10}\,dt = 2\sqrt{10} \approx 6.32$ cm.

4. (b) Let $f(x) = \ln x$. We'll determine $K = \max_{2 \le x \le 5} |f''(x)|$, then employ the Trapezoidal error estimate.

   - Now $f'(x) = \frac{1}{x} = x^{-1}$ and $f''(x) = -x^{-2}$. Thus $K = \max_{2\le x\le 5}\left(\frac{1}{x^2}\right) = \frac{1}{4}$.
   - Therefore,
   $$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{\frac{1}{4}(5-2)^3}{12(4)^2} = \frac{27}{3(4)^4} = \frac{9}{256}.$$

5. (a) The surface area is given by
$$S = \int 2\pi r\,ds = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^3 2\pi x \sqrt{1 + \left(\frac{1}{x}\right)^2}\,dx.$$





6. (c) The improper integral converges to $\pi/4$.
$$\begin{aligned} \int_1^\infty \frac{1}{x^2+1}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx \\ &= \lim_{t\to\infty} \left(\tan^{-1} x\right)\Big|_1^t \\ &= \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 1\right) \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

7. (a) The *slope of the curve* at a point (x, y) is x^3y^2 and the *curve passes through* $(0, \frac{1}{2})$. This gives us a separable differential equation $dy/dx = x^3y^2$ and an initial condition $y(0) = \frac{1}{2}$. Solve this initial value problem.
$$\begin{aligned} y^{-2}\,dy &= x^3\,dx \\ -\frac{1}{y} = -y^{-1} &= \tfrac{1}{4}x^4 + C \\ -2 &= C \quad \text{[Substitute the initial condition.]} \\ -\frac{1}{y} &= \tfrac{1}{4}x^4 - 2 \\ y &= \frac{1}{2 - \frac{1}{4}x^4} = \frac{4}{8 - x^4} \end{aligned}$$

8. (d) We have

$$0 \le \int_0^\infty \frac{1}{\sqrt{x}+e^{10x}}\,dx \le \int_0^\infty \frac{1}{e^{10x}}\,dx = \int_0^\infty e^{-10x}\,dx$$
$$= \lim_{t\to\infty}\int_0^t e^{-10x}\,dx$$
$$= \lim_{t\to\infty}\left(-\tfrac{1}{10}e^{-10x}\right)\Big|_0^t$$
$$= \lim_{t\to\infty}\left(-\tfrac{1}{10}e^{-10t}+\tfrac{1}{10}\right) = \frac{1}{10}.$$

Therefore, $\int_0^\infty \frac{1}{\sqrt{x}+e^{10x}}$ converges to a value $L \le \frac{1}{10}$.

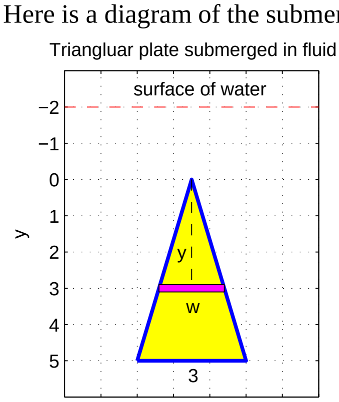
9. (d) With $y = \frac{2}{3}(x-1)^{3/2}$, $1 \le x \le 4$, the arc length is

$$\begin{aligned} L &= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_1^4 \sqrt{1+\left(\sqrt{x-1}\right)^2}\,dx \\ &= \int_1^4 x^{1/2}\,dx \\ &= \tfrac{2}{3}x^{3/2}\Big|_1^4 = \frac{16}{3}-\frac{2}{3} = \frac{14}{3}. \end{aligned}$$

10. (b) The step size is $h = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4}$. Partition points are $\left\{1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\right\}$. The Trapezoidal Rule gives
T_n = step size × (avg of endpt func vals + $\sum$ interior func vals)

$$\begin{aligned} \textit{or } T_4 &= \tfrac{1}{2}h\left(f(1)+2f\left(\tfrac{5}{4}\right)+2f\left(\tfrac{3}{2}\right)+2f\left(\tfrac{7}{4}\right)+f(2)\right) \\ &= \frac{1}{8}\left(\ln 1 + 2\ln\frac{5}{4} + 2\ln\frac{3}{2} + 2\ln\frac{7}{4} + \ln 2\right) \\ &= \frac{1}{8}\left(\ln 2 + 2\ln\frac{5}{4} + 2\ln\frac{3}{2} + 2\ln\frac{7}{4}\right) \end{aligned}$$

11. Here is a diagram of the submerged plate.

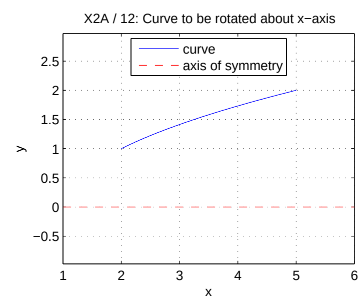


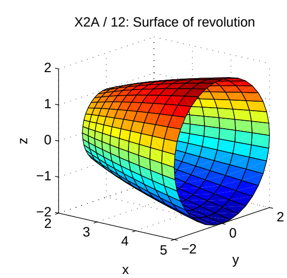
- Via similar triangles we have $3/5 = w/y$; so $w = \frac{3}{5}y$.
- A slice of the plate is at variable depth z. Build up your integral step-by-step: P, dA, dF, $F = \int dF$.

$$\begin{aligned} P &= \delta z = \delta\,(y-(-2)) = \delta\,(y+2) \\ dA &= w\,dy = \tfrac{3}{5}y\,dy \\ dF &= P\,dA = \tfrac{3}{5}\delta y\,(y+2)\,dy \\ F &= \tfrac{3}{5}\delta\int_0^5 y^2+2y\,dy \\ &= \tfrac{3}{5}\delta\left(\tfrac{1}{3}y^3+y^2\right)\Big|_0^5 \\ &= \tfrac{3}{5}(60)\left(\tfrac{125}{3}+25\right) = 2400 \text{ lb.} \end{aligned}$$

12. The surface area is $S = \int 2\pi r\,ds$ or

$$\begin{aligned} \int_c^d 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy &= \int_1^2 2\pi y\sqrt{1+(2y)^2}\,dy \\ &= \pi\int_1^2\left(1+4y^2\right)^{1/2}2y\,dy \\ &= \tfrac{\pi}{4}\left(\tfrac{2}{3}\right)\left(1+4y^2\right)^{3/2}\Big|_1^2 \\ &= \frac{\pi}{6}\left(17\sqrt{17}-5\sqrt{5}\right) \approx 30.85 \text{ cm}^2. \end{aligned}$$





13. The density ρ is constant. First compute the mass

$$m = \iint_D \rho\,dA = \rho\iint_D dA = \rho A = \rho\left(\tfrac{1}{4}\pi\,(2)^2\right) = \rho\pi$$

then compute $\bar{x}$, the x-coordinate of the centroid.

$$\begin{aligned} \bar{x} &= \frac{1}{m}\iint_D \rho x\,dA \qquad \text{[STEPS folks here; others below.]} \\ &= \frac{1}{\rho\pi}\int_0^2\int_0^{\sqrt{4-x^2}} \rho x\,dy\,dx \\ &= \frac{1}{\pi}\int_0^2 xy\Big|_{y=0}^{y=\sqrt{4-x^2}}\,dx \quad [\rho \text{ is constant}] \\ \bar{x} &= \frac{1}{\pi}\int_0^2\left(4-x^2\right)^{1/2}x\,dx \quad \left[\text{i.e., } \frac{1}{A}\int_a^b x f(x)\,dx\right] \\ &= \left(-\tfrac{1}{2}\right)\left(\tfrac{1}{\pi}\right)\left(\tfrac{2}{3}\right)\left(4-x^2\right)^{3/2}\Big|_0^2 \\ &= 0-\left(-\frac{8}{3\pi}\right) = \frac{8}{3\pi} \end{aligned}$$

14. (a) Factor to see that the differential equation is separable.

$$\begin{aligned}
\frac{dy}{dt} &= 1+t-yt-y=(t+1)-y(t+1)\\
\frac{dy}{dt} &= (t+1)(1-y)=-(t+1)(y-1)\\
\frac{1}{y-1}\,dy &= -(t+1)\,dt\\
\ln|y-1| &= -\tfrac{1}{2}(t+1)^2+A\\
|y-1| &= e^{A-(t+1)^2/2}=e^{A}e^{-(t+1)^2/2}\\
\pm(y-1) &= Be^{-(t+1)^2/2}\\
y &= 1+Ce^{-(t+1)^2/2}
\end{aligned}$$

(b) The differential equation is already in standard linear form (SLF): $y'-2y=x$. An integrating factor is $\mu=\exp\left(\int -2\,dx\right)=e^{-2x}$. Multiply the SLF by μ and solve.

$$\begin{aligned}
e^{-2x}y'-2e^{-2x}y &= xe^{-2x}\\
\left(e^{-2x}y\right)' &= xe^{-2x}\\
e^{-2x}y &= \left(-\tfrac{1}{2}x-\tfrac{1}{4}\right)e^{-2x}+C\\
y &= -\tfrac{1}{2}x-\tfrac{1}{4}+Ce^{2x}\\
1=y(0) &= -\tfrac{1}{4}+C \quad \text{[initial condition]}\\
C &= \tfrac{5}{4}\\
y &= -\tfrac{1}{2}x-\tfrac{1}{4}+\tfrac{5}{4}e^{2x}
\end{aligned}$$

In the third step we integrated by parts. With $u=x$ and $dv=e^{-2x}\,dx$ we have $du=dx$ and $v=-\frac{1}{2}e^{-2x}$. Therefore,

$$\begin{aligned}
\int xe^{-2x}\,dx &= -\frac{1}{2}xe^{-2x}+\int\frac{1}{2}e^{-2x}\,dx\\
&= \left(-\frac{1}{2}x-\frac{1}{4}\right)e^{-2x}+C
\end{aligned}$$

15. Let $y=y(t)$ be the amount of salt in the tank at time t. Since the tank initially contains pure water, we have $y(0)=0$ kg of salt in the tank at the start. The classical balance law gives

$$\begin{aligned}
\frac{dy}{dt} &= \text{rate in} - \text{rate out}\\
\frac{dy}{dt} &= \left(10\,\tfrac{\text{L}}{\text{min}}\times\tfrac{1}{2}\,\tfrac{\text{kg}}{\text{L}}\right)-\left(10\,\tfrac{\text{L}}{\text{min}}\times\tfrac{y\,\text{kg}}{1000\,\text{L}}\right)\\
\frac{dy}{dt} &= 5-\tfrac{1}{100}y \qquad \tfrac{\text{kg}}{\text{min}}\\
y'+\tfrac{1}{100}y &= 5.
\end{aligned}$$

The differential equation is both separable and linear. Let's find an integrating factor for the standard linear form (SLF) in the last equation above: $\mu=\exp\left(\int\frac{1}{100}\,dt\right)=e^{t/100}$.

Multiply the SLF by μ and solve.

$$\begin{aligned}
e^{t/100}y'+\tfrac{1}{100}e^{t/100}y &= 5e^{t/100}\\
\left(e^{t/100}y\right)' &= 5e^{t/100}\\
e^{t/100}y &= 500e^{t/100}+C\\
y &= 500+Ce^{-t/100}\\
0=y(0) &= 500+C \quad \text{[initial condition]}\\
C &= -500\\
y &= 500-500e^{-t/100}\\
y(10) &= 500-500e^{-10/100}\approx 47.58\text{ kg}
\end{aligned}$$