Name____________________ ID___________

1-11	/55
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Total	/105

MATH 171 Final Exam Spring 2004

Sections 502 Solutions P. Yasskin

On the front of the Blue Book, on the Scantron and on this sheet
write your Name, your University ID and "Final Exam."
On the front of the Blue Book copy the Grading Grid shown at the right.
Enter your Multiple Choice answers on the Scantron
and CIRCLE them on this sheet.

Multiple Choice: (5 points each. No part credit.)

1. For what value(s) of p are the vectors $\vec{a} = (3,p)$ and $\vec{b} = (4,6)$, perpendicular?

a. 2 or -2 only

b. 2 only

c. -2 only correctchoice

d. $\frac{1}{2}$ only

e. $-\frac{1}{2}$ only

$\vec{a}$ and $\vec{b}$ are perpendicular iff $\vec{a} \cdot \vec{b} = 0$. In this case, $\vec{a} \cdot \vec{b} = 12 + 6p = 0$. So $p = -2$.

2. For what value of b does $\lim\limits_{x\to 2} f(x)$ exist if $f(x) = \begin{cases} x+3 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ x^2 + b & \text{if } x > 2 \end{cases}$

a. 1 correctchoice

b. 2

c. 3

d. 4

e. No values of b.

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (x+3) = 5$ $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (x^2 + b) = 4 + b$

So $\lim\limits_{x\to 2} f(x)$ exist iff $4 + b = 5$, i.e. $b = 1$.

3. Compute $\lim\limits_{x\to 3} \dfrac{x^2-4x+3}{x^2-9}$

a. $\frac{1}{6}$

b. $\frac{1}{3}$ correctchoice

c. $\frac{1}{2}$

d. $\frac{2}{3}$

e. $\frac{5}{6}$

$$\lim_{x\to 3} \frac{x^2-4x+3}{x^2-9} = \lim_{x\to 3} \frac{(x-3)(x-1)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{(x-1)}{(x+3)} = \frac{2}{6} = \frac{1}{3}$$

4. Compute $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3}$

a. $-\frac{1}{6}$ correctchoice

b. $-\frac{1}{3}$

c. 0

d. $\frac{1}{3}$

e. undefined

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} \overset{l'H}{=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} \overset{l'H}{=} \lim_{x\to 0} \frac{-\sin x}{6x} = \frac{-1}{6}$$

5. As $x \to \infty$, the function $f(x) = \sqrt{x^2+5x} - \sqrt{x^2+2x}$ has a horizontal asymptote at

a. $-\frac{3}{2}$

b. $-\frac{1}{2}$

c. 0

d. $\frac{1}{2}$

e. $\frac{3}{2}$ correctchoice

$$\lim_{x\to\infty}\left(\sqrt{x^2+5x} - \sqrt{x^2+2x}\right) = \lim_{x\to\infty}\left(\sqrt{x^2+5x} - \sqrt{x^2+2x}\right) \cdot \frac{\sqrt{x^2+5x} + \sqrt{x^2+2x}}{\sqrt{x^2+5x} + \sqrt{x^2+2x}}$$

$$= \lim_{x\to\infty} \frac{(x^2+5x)-(x^2+2x)}{\sqrt{x^2+5x} + \sqrt{x^2+2x}} = \lim_{x\to\infty} \frac{3x}{\sqrt{x^2+5x} + \sqrt{x^2+2x}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{3}{\sqrt{1+\frac{5}{x}} + \sqrt{1+\frac{2}{x}}} = \frac{3}{2}$$

6. If $f(x) = \ln(x^2 + x)$ then $f'(2) =$

a. $\frac{1}{6}$

b. $\frac{1}{3}$

c. $\frac{1}{2}$

d. $\frac{2}{3}$

e. $\frac{5}{6}$ correctchoice

$f'(x) = \frac{2x+1}{x^2+x}$ $f'(2) = \frac{5}{6}$

7. If Pete is walking up a hill whose slope is 0.2 and his horizontal velocity is $\frac{dx}{dt} = 6$ mi/hr, what is his vertical velocity, $\frac{dy}{dt}$?

a. 30 mi/hr

b. $0.03\bar{3}$ mi/hr

c. $0.83\bar{3}$ mi/hr

d. 1.2 mi/hr correctchoice

e. 6.2 mi/hr

By chain rule, $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = 0.2 \cdot 6 = 1.2$ mi/hr.

8. $x = 2$ is a critical point of the function $f(x) = \frac{1}{4}x^4 - 2x^3 + 6x^2 - 8x.$
By the Second Derivative Test, $x = 2$ is

a. a local minimum.

b. a local maximum.

c. an inflection point.

d. The Second Derivative Test FAILS. correctchoice

$f'(x) = x^3 - 6x^2 + 12x - 8$ $f''(x) = 3x^2 - 12x + 12$ $f''(2) = 12 - 24 + 12 = 0$ Test Fails.

9. If a rocket starts at $x(0) = 0$ m, with velocity $v(0) = 1$ m/sec, and accelerates at $a(t) = 4e^{-2t}$ m/sec^2, what is its position at $t = 1$ sec?

a. $2 - 16e^{-2}$

b. $2 + e^{-2}$ correctchoice

c. 3

d. $1 + e^{-2}$

e. $2 + 16e^{-2}$

$v(t) = -2e^{-2t} + C \quad v(0) = -2 + C = 1 \quad C = 3 \quad v(t) = -2e^{-2t} + 3$

$x(t) = e^{-2t} + 3t + D \quad x(0) = 1 + D = 0 \quad D = -1 \quad x(t) = e^{-2t} + 3t - 1 \quad x(1) = e^{-2} + 3 - 1 = 2 + e^{-2}$

10. Compute $\displaystyle\int_{1/2}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$

a. $\frac{\pi}{12}$

b. $\frac{\pi}{6}$

c. $\frac{\pi}{4}$

d. $\frac{\pi}{3}$ correctchoice

e. $\frac{\pi}{2}$

$$\int_{1/2}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \Big[\arcsin x\Big]_{1/2}^{1} = \arcsin 1 - \arcsin\frac{1}{2} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

11. Compute $\displaystyle\int_{0}^{\pi} e^{\cos x}\sin x\,dx$

a. 0

b. $\frac{1}{e} - e$

c. $e - \frac{1}{e}$ correctchoice

d. $-\frac{1}{e}$

e. $-e$

$u = \cos x \quad du = -\sin x\,dx$

$$\int_{0}^{\pi} e^{\cos x}\sin x\,dx = -\int e^u\,du = -e^u = \Big[-e^{\cos x}\Big]_0^{\pi} = -e^{-1} + e^1 = e - \frac{1}{e}$$

Work Out: (10 points each. Part credit possible.)

Start each problem on a new page of the Blue Book. Number the problem. Show all work.

12. Find the equation of the line tangent to $y = x^2$ at the general point $x = a$.
For what value(s) of a does the tangent line pass through the point $(3, 8)$?

$f(x) = x^2 \quad f(a) = a^2 \quad f'(x) = 2x \quad f'(a) = 2a$

$y = f_{\tan}(x) = f(a) + f'(a)(x-a) = a^2 + 2a(x-a) = 2ax - a^2$

$(3, 8)$ lies on the tangent line if $8 = 2a3 - a^2$, or $a^2 - 6a + 8 = 0$, or $(a-2)(a-4) = 0$

So $a = 2$ or $a = 4$.

13. The area of a rectangle is held constant at 36 cm^2 while the length and width are changing. If the length is currently 3 cm and is increasing at 2 cm/min, what is the width, is it increasing or decreasing and at what rate? Write your answer using sentences.

Let l be the length and w be the width of the rectangle.

Then the area is held constant at $A = lw = 36$. Solving for the width we find $w = \dfrac{36}{l}$.

Differentiating and using the chain rule, we find $\dfrac{dw}{dt} = -\dfrac{36}{l^2}\dfrac{dl}{dt}$.

Currently, $l = 3$ and $\dfrac{dl}{dt} = 2$. So currently the width is $w = \dfrac{36}{3} = 12$ cm,

and it is changing at $\dfrac{dw}{dt} = -\dfrac{36}{3^2}2 = -8$ cm/min. So it is decreasing at 8 cm/min.

14. Determine exactly how many real solutions there are to the equation $x^{12} + x^4 + x^2 - 2 = 0$.
Use sentences and name any theorems you use.
Hint: Factor an x out of the derivative.

Let $f(x) = x^{12} + x^4 + x^2 - 2$. Then $f'(x) = 12x^{11} + 4x^3 + 2x = x(12x^{10} + 4x^2 + 2)$.

The quantity in parentheses is always positive.

So for $x > 0$ we have $f'(x) > 0$ and for $x < 0$ we have $f'(x) < 0$.

By the Mean Value Theorem, $f(x)$ is increasing for $x > 0$ and decreasing for $x < 0$.

So there can be at most one solution for $x > 0$ and at most one solution for $x < 0$.

We test some values:

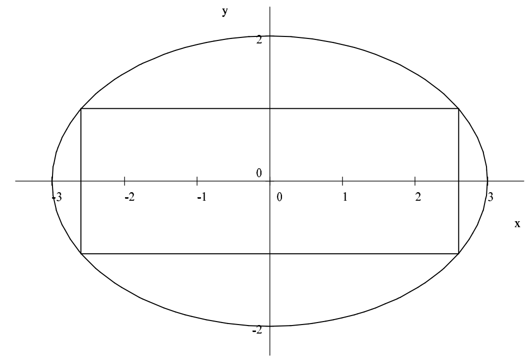
$$f(-1) = 1 + 1 + 1 - 2 = 1 \qquad f(0) = -2 \qquad f(1) = 1 + 1 + 1 - 2 = 1$$

Since $-2 < 0 < 1$, by the Intermediate Value Theorem,

there is at least one solution of $f(x) = 0$ on $[-1, 0]$ and at least one solution on $[0, 1]$.

Therefore, there are exactly 2 solutions.

15. Find the dimensions and area of the largest rectangle that can be inscribed in the ellipse $4x^2 + 9y^2 = 36$.



Maximize $A = 4xy$ subject to the constraint $4x^2 + 9y^2 = 36$.

Solve the constraint for $y = \frac{1}{3}\sqrt{36 - 4x^2}$ and substitute into the area:

Maximize $A = \frac{4}{3}x\sqrt{36 - 4x^2}$

$$A' = \frac{4}{3}\sqrt{36 - 4x^2} + \frac{4}{3}x\frac{-4x}{\sqrt{36 - 4x^2}} = 0 \quad \Rightarrow \quad \sqrt{36 - 4x^2} = \frac{4x^2}{\sqrt{36 - 4x^2}}$$

$$\Rightarrow \quad 36 - 4x^2 = 4x^2 \quad \Rightarrow \quad 36 = 8x^2 \quad \Rightarrow \quad x^2 = \frac{36}{8} = \frac{9}{2} \quad \Rightarrow \quad x = \frac{3}{\sqrt{2}}$$

$$y = \frac{1}{3}\sqrt{36 - 4x^2} = \frac{1}{3}\sqrt{36 - 4 \cdot \frac{9}{2}} = \frac{1}{3}\sqrt{18} = \sqrt{2}$$

The dimensions are $2x = \frac{6}{\sqrt{2}} = 3\sqrt{2}$ and $2y = 2\sqrt{2}$ and the area is

$A = 4xy = 4\frac{3}{\sqrt{2}}\sqrt{2} = 12$.

16. Use the Method of Riemann Sums with Right Endpoints to compute the integral $\int_2^7 8(x-2)^3\,dx$.

Use the F.T.C. only to check your answer.

Hints: $\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

$$\Delta x = \frac{7-2}{n} = \frac{5}{n} \qquad x_i = 2 + i\Delta x = 2 + \frac{5i}{n} \qquad f(x) = 8(x-2)^3 \qquad f(x_i) = 8\left(2 + \frac{5i}{n} - 2\right)^3 = \frac{1000i^3}{n^3}$$

$$\int_2^7 8(x-2)^3\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1000i^3}{n^3}\frac{5}{n} = \lim_{n\to\infty}\frac{5000}{n^4}\sum_{i=1}^{n} i^3 = \lim_{n\to\infty}\frac{5000}{n^4}\left(\frac{n(n+1)}{2}\right)^2$$

$$= \lim_{n\to\infty} 5000\left(\frac{n(n+1)}{2n^2}\right)^2 = \lim_{n\to\infty} 5000\left(\frac{(n+1)}{2n}\right)^2 = \lim_{n\to\infty} 5000\left(\frac{1}{2} + \frac{1}{2n}\right)^2 = \frac{5000}{4} = 1250$$

Check: $\int_2^7 8(x-2)^3\,dx = \left[2(x-2)^4\right]_2^7 = 2(5)^4 = 1250$.