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1-10	/50	12	/25
11	/ 5	13	/25
		Total	/105

MATH 171 Exam 1 Fall 2021

Sections 503 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Write $\langle 1,5\rangle$ as a linear combination of $\langle 2,3\rangle$ and $\langle 3,1\rangle$ or type "impossible" in both boxes.

$$\langle 1,5\rangle = \_\_\_\_\_\_\langle 2,3\rangle + \_\_\_\_\_\_\langle 3,1\rangle$$

Solution: Let $\langle 1,5\rangle = a\langle 2,3\rangle + b\langle 3,1\rangle$. Then $1 = 2a+3b$ and $5 = 3a+b$.
So $b = 5-3a$ and $1 = 2a+3(5-3a) = 15-7a$ or $7a = 14$ or $a = 2$.
Then $b = 5-3(2) = -1$ So $\langle 1,5\rangle = \underline{\ 2\ }\langle 2,3\rangle + \underline{\ -1\ }\langle 3,1\rangle$.

2. Find the angle between the vectors $\langle 2,3\rangle$ and $\langle 5,1\rangle$.

- **a**. $0°$
- **b**. $30°$
- **c**. $45°$ Correct Choice
- **d**. $60°$
- **e**. $90°$
- **f**. $120°$
- **g**. $135°$
- **h**. $150°$
- **i**. $180°$

Solution: Let $\vec{a} = \langle 2,3\rangle$ and $\vec{b} = \langle 5,1\rangle$. Then

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{10+3}{\sqrt{4+9}\sqrt{25+1}} = \frac{13}{\sqrt{13}\sqrt{26}} = \frac{1}{\sqrt{2}}$$

So $\theta = 45°$

3. Write $\vec{v} = \langle 10,5\rangle$ as the sum of two vectors $\vec{p}$ and $\vec{q}$ where $\vec{p}$ is parallel to $\vec{u} = \langle 3,4\rangle$ and $\vec{q}$ is perpendicular to $\vec{u}$.

$$\langle 10,5\rangle = \vec{p}+\vec{q}$$

where

$$\vec{p} = \langle\_\_\_\_\_\_,\_\_\_\_\_\_\rangle \quad \text{and} \quad \vec{q} = \langle\_\_\_\_\_\_,\_\_\_\_\_\_\rangle$$

Solution: $\vec{p} = \text{proj}_{\vec{u}}\vec{v} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}|^2}\vec{u} = \dfrac{30+20}{9+16}\langle 3,4\rangle = \dfrac{50}{25}\langle 3,4\rangle = \langle 6,8\rangle$

$\vec{q} = \text{proj}_{\perp\vec{u}}\vec{v} = \vec{v} - \text{proj}_{\vec{u}}\vec{v} = \langle 10,5\rangle - \langle 6,8\rangle = \langle 4,-3\rangle$

Check: $\vec{p}+\vec{q} = \langle 6,8\rangle + \langle 4,-3\rangle = \langle 10,5\rangle$

$\vec{p} = \langle 6,8\rangle$ is a multiple of $\vec{u} = \langle 3,4\rangle$ $\quad\vec{q}\cdot\vec{u} = \langle 4,-3\rangle\cdot\langle 3,4\rangle = 12-12 = 0$

4. Find the smallest interval with integer endpoints in which there is a solution of the equation $x^3 + 3x = 40$.
There is a solution in the interval $I = [\_\_\_\_\_\_, \_\_\_\_\_\_]$.

Solution: Let $f(x) = x^3 + 3x$. We plug in values:

$$f(0) = 0 \qquad f(1) = 4 \qquad f(2) = 14 \qquad f(3) = 36 \qquad f(4) = 76$$

Since $f(x)$ is continuous and $36 < 40 < 76$, the Intermediate Value Theorem guarantees there is a solution to $f(x) = 40$ in the interval $I = [3,4]$.

5. For the piecewise defined function $f(x) = \begin{cases} 5 & \text{for } x = 4 \\ 9 - x & \text{for } x > 4 \\ 2 + x & \text{for } x < 4 \end{cases}$ identify

$$f(4) = \_\_\_\_\_\_ \qquad \lim_{x \to 4^-} f(x) = \_\_\_\_\_\_ \qquad \lim_{x \to 4^+} f(x) = \_\_\_\_\_\_$$

Then enter T or F to say if each statement is true or false:

a. $f(x)$ is continuous ______ (TorF)

b. $f(x)$ is continuous from the right ______ (TorF)

c. $f(x)$ is continuous from the left ______ (TorF)

d. $\lim_{x \to 4} f(x)$ exists ______ (TorF)

Solution: $f(4) = \_\_\_5\_\_\_$ $\qquad \lim_{x \to 4^-} f(x) = \_\_\_6\_\_\_$ $\qquad \lim_{x \to 4^+} f(x) = \_\_\_5\_\_\_$

a. $f(x)$ is continuous ___F___

b. $f(x)$ is continuous from the right ___T___

c. $f(x)$ is continuous from the left ___F___

d. $\lim_{x \to 4} f(x)$ exists ____F___

6. Find the interval on which $g(x) = \dfrac{1}{\sqrt{4 - x^2}} + \sqrt{x}$ is continuous.

a. $-2 < x \le 0$

b. $-2 \le x \le 2$

c. $2 < x < \infty$

d. $0 \le x < 2$ Correct Choice

e. $0 < x \le 2$

Solution: Since $\sqrt{4 - x^2}$ is in the denominator, it cannot be 0. Further, $4 - x^2$ must be non-negative. So $4 - x^2 > 0$ or $x^2 < 4$ or $-2 < x < 2$. Also $x \ge 0$ for $\sqrt{x}$ to be defined. Together, they say $0 \le x < 2$.

7. Find the horizontal asymptotes for $g(x) = \dfrac{5 \cdot 3^x + 4}{3^x + 2}$.

As $x \to +\infty$, the horizontal asymptote is $y =$ ______

As $x \to -\infty$, the horizontal asymptote is $y =$ ______

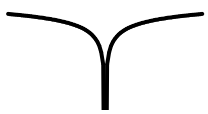
Solution: As $x \to \infty$, we know $3^x \to \infty$ but $3^{-x} \to 0$. So:

$$\lim_{x\to\infty} \frac{5 \cdot 3^x + 4}{3^x + 2} = \lim_{x\to\infty} \frac{5 + 4 \cdot 3^{-x}}{1 + 2 \cdot 3^{-x}} = \frac{5 + 4 \cdot 0}{1 + 2 \cdot 0} = 5 \qquad y = 5$$

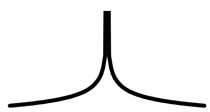
$$\lim_{x\to-\infty} \frac{5 \cdot 3^x + 4}{3^x + 2} = \frac{5 \cdot 0 + 4}{0 + 2} = 2 \qquad y = 2$$

8. The function $f(x) = \dfrac{x - 4}{(x-2)^2}$ has a vertical asymptote at $x = 2$. Near $x = 2$, its graph looks like:

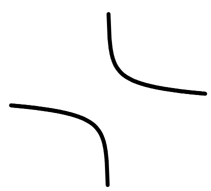
a.



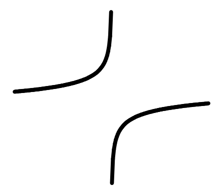
b.



c.



d.



e None of these

Solution:

$$\lim_{x\to 2^-} \frac{x-4}{(x-2)^2} = \frac{2^- - 4}{(2^- - 2)^2} = \frac{-2}{(0^-)^2} = -\infty \qquad \text{down on left}$$

$$\lim_{x\to 2^+} \frac{x-4}{(x-2)^2} = \frac{2^+ - 4}{(2^+ - 2)^2} = \frac{-2}{(0^+)^2} = -\infty \qquad \text{down on right}$$

Answer a

9. Find the average velocity between $t_1 = 1$ and $t_2 = 1.1$ if the position is $x(t) = t^2$.

a. 12.01

b. 12.1

c. 2

d. 2.01

e. 2.1 Correct Choice

Solution: $v_{\text{ave}} = \dfrac{x(t_2) - x(t_1)}{t_2 - t_1} = \dfrac{x(1.1) - x(1)}{1.1 - 1} = \dfrac{1.1^2 - 1}{.1} = \dfrac{1.21 - 1}{.1} = \dfrac{.21}{.1} = 2.1$

10. Find the tangent line to the curve $y = \frac{1}{x^2}$ at $x = 2$. It can be written in slope intercept form as $y = mx + b$, where

$m =$ ______ and $b =$ ______

Solution: Let $f(x) = \frac{1}{x^2}$.

$$m = f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{\frac{1}{(2+h)^2}-\frac{1}{4}}{h} = \lim_{h\to 0}\frac{1}{h}\left[\frac{4-(2+h)^2}{4(2+h)^2}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{4-(4+4h+h^2)}{4(2+h)^2}\right] = \lim_{h\to 0}\frac{1}{h}\left[\frac{-(4h+h^2)}{4(2+h)^2}\right] = \lim_{h\to 0}\left[\frac{-(4+h)}{4(2+h)^2}\right] = \frac{-(4)}{4(2)^2} = \frac{-1}{4}$$

Alternatively, since $f(x) = x^{-2}$, the power rule says $f'(x) = -2x^{-3} = \frac{-2}{x^3}$. So:

$$m = f'(2) = \frac{-1}{4}$$

To find the equation of the line we use the point-slope form:

$$y = y_0 + m(x - x_0) = f(2) + f'(2)(x-2)$$
$$= \frac{1}{4} - \frac{1}{4}(x-2) = -\frac{1}{4}x + \frac{1}{4} + \frac{1}{2} = -\frac{1}{4}x + \frac{3}{4}$$

So:

$$b = \frac{3}{4}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (5 points) Write out the definition of the statement $\lim_{x\to 4} x^3 = 64$.

Your answer must consist of words, phrases and formulas from the following list:

For	and	$\varepsilon > 0$	$\delta > 0$
such that	or	$\lvert x-4\rvert < \varepsilon$	$\lvert x-4\rvert < \delta$
there exists	if	$0 < \lvert x-4\rvert < \varepsilon$	$0 < \lvert x-4\rvert < \delta$
there does not exist	then	$\lvert x^3-64\rvert < \varepsilon$	$\lvert x^3-64\rvert < \delta$
some	all	$0 < \lvert x^3-64\rvert < \varepsilon$	$0 < \lvert x^3-64\rvert < \delta$

$\lim_{x\to 4} x^3 = 64$ means:

Solution: For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x-4| < \delta$ then $|x^3 - 64| < \varepsilon$.

12. (25 points) Compute each of the following limits.

a. $\lim\limits_{k\to 4} \dfrac{k-4}{k^2-k-12} =$

Solution: Factor and Cancel:

$$\lim_{k\to 4} \frac{k-4}{k^2-k-12} = \lim_{k\to 4} \frac{k-4}{(k-4)(k+3)} = \lim_{k\to 4} \frac{1}{k+3} = \frac{1}{7}$$

b. $\lim\limits_{x\to 5} \dfrac{(x-10)^2-25}{x-5} =$

Solution: Expand and Cancel:

$$\lim_{x\to 5} \frac{(x-10)^2-25}{x-5} = \lim_{x\to 5} \frac{(x^2-20x+100)-25}{x-5} = \lim_{x\to 5} \frac{x^2-20x+75}{x-5}$$
$$= \lim_{x\to 5} \frac{(x-5)(x-15)}{x-5} = \lim_{x\to 5}(x-15) = -10$$

c. $\lim\limits_{x\to 4} \dfrac{x-4}{\sqrt{12+x}-\sqrt{20-x}} =$

Solution: Multiply by the Conjugate

$$\lim_{x\to 4} \frac{x-4}{\sqrt{12+x}-\sqrt{20-x}} = \lim_{x\to 4} \frac{x-4}{\sqrt{12+x}-\sqrt{20-x}} \cdot \frac{\sqrt{12+x}+\sqrt{20-x}}{\sqrt{12+x}+\sqrt{20-x}}$$
$$= \lim_{x\to 4} \frac{(x-4)\left(\sqrt{12+x}+\sqrt{20-x}\right)}{(12+x)-(20-x)} = \lim_{x\to 4} \frac{(x-4)\left(\sqrt{12+x}+\sqrt{20-x}\right)}{2x-8}$$
$$= \lim_{x\to 4} \frac{\sqrt{12+x}+\sqrt{20-x}}{2} = \frac{\sqrt{16}+\sqrt{16}}{2} = \frac{8}{2} = 4$$

d. $\lim\limits_{x\to\infty}\left(x-\dfrac{x^2+3}{x+4}\right) =$

Solution: Put the Limit over a Common Denominator

$$\lim_{x\to\infty}\left(x-\frac{x^2+3}{x+4}\right) = \lim_{x\to\infty}\left(\frac{x(x+4)}{x+4}-\frac{x^2+3}{x+4}\right) = \lim_{x\to\infty}\left(\frac{x^2+4x-x^2-3}{x+4}\right)$$
$$= \lim_{x\to\infty}\left(\frac{4x-3}{x+4}\right) = \lim_{x\to\infty}\left(\frac{4-\frac{3}{x}}{1+\frac{4}{x}}\right) = 4$$

e. $\lim\limits_{\theta\to 0} \dfrac{1-\cos^4\theta}{\theta^2} =$

Solution: Factor and use the limit $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.

$$\lim_{\theta\to 0} \frac{1-\cos^4\theta}{\theta^2} = \lim_{\theta\to 0} \frac{(1-\cos^2\theta)(1+\cos^2\theta)}{\theta^2} = \lim_{\theta\to 0} \frac{\sin^2\theta(1+\cos^2\theta)}{\theta^2}$$
$$= \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^2(1+\cos^2\theta) = (1)^2(1+1) = 2$$

13. (25 points) Compute the derivative of each of the following functions.

a. $f(x) = 5x^4 - 3x^2 + 7x - \dfrac{2}{x^3}$

Solution: Note $\dfrac{1}{x^3} = x^{-3}$. By the sum, constant multiple and power rules:

$$f'(x) = 5 \cdot 4x^3 - 3 \cdot 2x^1 + 7 \cdot 1x^0 - 2(-3)x^{-4} = 20x^3 - 6x^1 + 7x^0 + \frac{6}{x^4}$$

b. $g(y) = y^3 \cos(y)$

Solution: By the product and trig rules:

$$g'(y) = \frac{d}{dy}(y^3)\cos(y) + y^3 \frac{d}{dy}[\cos(y)] = 3y^2\cos(y) + y^3[-\sin(y)] = 3y^2\cos(y) - y^3\sin(y)$$

c. $h(t) = \dfrac{\sin(t)}{t}$

Solution: By the quotient and trig rules:
(bottom times the derivative of the top tminus the top times the derivative of the bottom over the bottom squared.)

$$h'(t) = \frac{t\frac{d}{dt}[\sin(t)] - \sin(t)\frac{d}{dt}(t)}{t^2} = \frac{t\cos(t) - \sin(t)}{t^2}$$

d. $k(x) = 2x^e + 3e^x$

Solution: By the sum, constant multiple, power rules and exponential rules:

$$k'(x) = 2\frac{d}{dx}(x^e) + 3\frac{d}{dx}(e^x) = 2ex^{e-1} + 3e^x$$

e. If $f(x) = \dfrac{p(x) + q(x)}{r(x)}$, find $f'(1)$, given that

$$p(1) = 7, \quad p'(1) = 6, \quad q(1) = 5, \quad q'(1) = 4, \quad r(1) = 3, \quad r'(1) = 2$$

Solution: By the quotient and sum rules:

$$f'(x) = \frac{r(x)\frac{d}{dx}[p(x) + q(x)] - [p(x) + q(x)]\frac{d}{dx}[r(x)]}{[r(x)]^2} = \frac{r(x)[p'(x) + q'(x)] - [p(x) + q(x)]r'(x)}{[r(x)]^2}$$

$$f'(1) = \frac{r(1)[p'(1) + q'(1)] - [p(1) + q(1)]r'(1)}{[r(1)]^2} = \frac{3[6+4] - [7+5]2}{[3]^2} = \frac{30 - 24}{9} = \frac{2}{3}$$