Name________________________UIN__________________

1-10	/57	13	/10
11	/15	14	/15
12	/10	Total	/107

MATH 171 Exam 3 Fall 2021

Sections 503 Solutions P. Yasskin

Multiple Choice: (5 points each, unless indicated. No part credit.)

1. If $L = \sqrt{x^2 + y^2}$, find $\dfrac{dL}{dt}$ given that

$$x = 4 \qquad y = 3 \qquad \frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 2$$

$\dfrac{dL}{dt} =$ ____________________

Solution: By chain rule,

$$\frac{dL}{dt} = \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right) = \frac{1}{\sqrt{4^2+3^2}}(4(1)+3(2)) = \frac{1}{5}(10) = \underline{\ 2\ }.$$

2. Find the horizontal asymptotes of the function $f(x) = \dfrac{6e^x + 12}{3e^x - 4}$.

a. $y = 2$ only

b. $y = -3$ only

c. $y = \ln\left(\frac{4}{3}\right)$ only

d. $y = 2$ and $y = -3$ only Correct Choice

e. $y = 2$ and $y = \ln\left(\frac{4}{3}\right)$ only

f. $y = -3$ and $y = \ln\left(\frac{4}{3}\right)$ only

g. $y = 2$, $y = -3$ and $y = \ln\left(\frac{4}{3}\right)$

h. None

Solution: $\lim\limits_{x\to\infty}\dfrac{6e^x+12}{3e^x-4} = \lim\limits_{x\to\infty}\dfrac{6+12e^{-x}}{3-4e^{-x}} = \dfrac{6+0}{3-0} = 2 \qquad \lim\limits_{x\to-\infty}\dfrac{6e^x+12}{3e^x-4} = \dfrac{0+12}{0-4} = -3$

Note: $x = \ln\left(\frac{4}{3}\right)$ is a *vertical* asymptote.

3. Find the x-coordinate(s) on the graph of $f(x) = x^4 + 4x^3 + 5x^2 + 3x + 1$ where the **curvature** is a minimum.
Enter one or more numbers separated by commas, no spaces.

$x =$ ____________________

Solution: The curvature is the second derivative.

$f'(x) = 4x^3 + 12x^2 + 10x + 3 \qquad$ curvature $= f''(x) = 12x^2 + 24x + 10$

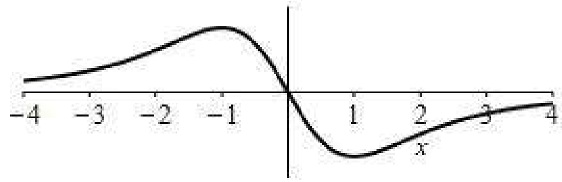
To find where the curvature is a minimum, we set its derivative equal to 0 and solve for the critical points.

curvature$' = f'''(x) = 24x + 24 = 0$ at $x = \underline{\ -1\ }$.

To check it is a minimum, we substitute the critical point into the second derivative.

curvature$'' = f''''(x) = 24 > 0$ So it is a minimum.

4. (10 points) This is the graph of f', i.e. the derivative of f.



a. Identify the interval(s) where f is increasing.
Enter one or more intervals separated by commas, no spaces.
Include finite endpoints in the intervals. All numbers are integers. Type infinity for ∞.

Increasing on ___________________________.

Solution: The function is increasing when its derivative is positive which occurs on the interval $(-\infty,0]$.

b. Identify the interval(s) where f is concave up.
Enter one or more intervals separated by commas, no spaces.
Include finite endpoints in the intervals. All numbers are integers. Type infinity for ∞.

Concave Up on ___________________________.

Solution: The function is concave up when its second derivative is positive which means the first derivative is increasing which occurs on the intervals $(-\infty,-1]$ and $[1,\infty)$.

5. The point $x = 1$ is a critical point of the function $f(x) = x^3 - 3x^2 + 3x$.
Then the Second Derivative Test implies $x = 1$ is a

a. Local Minimum
b. Local Maximum
c. Inflection Point
d. Test Fails Correct Choice

Solution: $f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2 = 0$ at $x = 1$.
$f''(x) = 6x - 6$ and so $f''(1) = 6 - 6 = 0$. The Second Derivative Test FAILS.

6. (6 points) Find the locations of the absolute maximum and minimum of $f(x) = x^3 - 6x^2 + 9x$ on the interval $[-2,2]$.

The absolute minimum occurs at $x =$ ___________________.

The absolute maximum occurs at $x =$ ___________________.

Solution: $f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3) = 0$.
The critical point $x = 3$ is not in the interval $[-2,2]$ so we ignore it.
We evaluate the function at the other critical point $x = 1$ and the endpoints:
$f(1) = 1 - 6 + 9 = 4$ $\quad f(-2) = (-2)^3 - 6(-2)^2 + 9(-2) = -8 - 24 - 18 = -50$
$f(2) = (2)^3 - 6(2)^2 + 9(2) = 8 - 24 + 18 = 2$
So the minimum occurs at $x = -2$ and the maximum occurs at $x = 1$.

7. Use a Riemann Sum with 4 equal intervals and right endpoints to approximate $\int_1^9 (x^2+1)\,dx$.

$\int_1^9 (x^2+1)\,dx \approx$ ____________________

Solution: The function is $f(x) = x^2 + 1$. The width of the intervals is $\Delta x = \frac{9-1}{4} = 2$.
The right endpoints are $3,\ 5,\ 7$ and 9.
The function values are $f(3) = 10,\ f(5) = 26,\ f(7) = 50$ and $f(9) = 82$.
So the Riemann sum is $\sum_{n=1}^{4} f(x_i)\Delta x = (10+26+50+82)2 = 336$.
$\int_1^9 (x^2+1)\,dx \approx$ __336__.

8. (6 points) A rocket starts with an initial height $y(0) = 8$ m and an initial velocity of $v(0) = 0\ \frac{\text{m}}{\text{sec}}$.
If its acceleration is $a(t) = 5e^t$, find its velocity and height at time $t = \ln 3$.
Put an integer in each blank.

$v(\ln 3) =$ _______________ + _______________ $\ln 3$

$y(\ln 3) =$ _______________ + _______________ $\ln 3$

Solution: $\frac{dv}{dt} = a(t) = 5e^t \qquad v(t) = 5e^t + C \qquad v(0) = 5 + C = 0 \qquad C = -5 \qquad v(t) = 5e^t - 5$
$\frac{dy}{dt} = v(t) = 5e^t - 5 \qquad y(t) = 5e^t - 5t + K \qquad y(0) = 5 + K = 8 \qquad K = 3 \qquad y(t) = 5e^t - 5t + 3$
$v(\ln 3) = 5e^{\ln 3} - 5 = 15 - 5 =$ __$10 + 0\ln 3$__ $\qquad y(\ln 3) = 5e^{\ln 3} - 5\ln 3 + 3 =$ __$18 - 5\ln 3$__

9. Find the area between the curves $y = 3x^2$ and $y = 6x$.

$A =$ ____________________

Solution: To find the intersections, we equate the functions:
$3x^2 = 6x \qquad 3x^2 - 6x = 3x(x-2) = 0$ So they intersect at $x = 0, 2$.
$3x^2$ bends upward while $6x$ is a straight line. So $3x^2$ is on the bottom and $6x$ is on top.
$A = \int_0^2 (6x - 3x^2)\,dx = \left[3x^2 - x^3\right]_0^2 = (12-8) - 0 =$ __4__.

10. Find the mass of a bar of length π cm, if its linear density is $\delta = 1 + \sin x$
where x is measured from one end.
Put an rational number in each blank.

$M =$ _______________ + _______________ π

Solution: $M = \int_0^\pi (1 + \sin x)\,dx = \left[x - \cos x\right]_0^\pi = (\pi - -1) - (0 - 1) = \pi + 2$

Work Out: (Points indicated. Part credit possible. Show all work.)

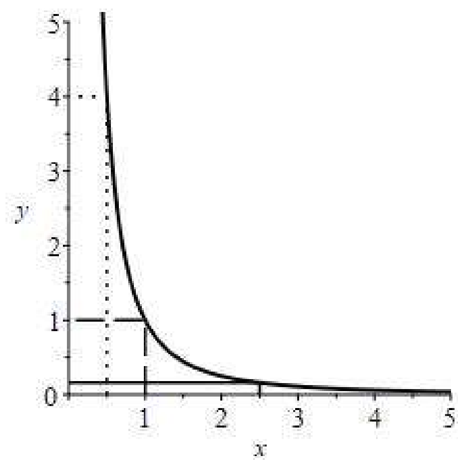
11. (15 points) A cone with the vertex at the bottom has height $H = 6$ cm and radius $R = 3$ cm at the top. It is being filled with water at the rate of $\dfrac{dV}{dt} = 2\pi \dfrac{\text{cm}^3}{\text{sec}}$.

How fast is the height of the water increasing when it is 4 cm deep?

HINT: The volume of a cone is $V = \frac{1}{3}\pi r^2 h.$

Solution: Let h and r be the height and radius of the water.
By similar triangles, $\dfrac{r}{h} = \dfrac{R}{H} = \dfrac{3}{6} = \dfrac{1}{2}$. So $r = \dfrac{1}{2}h$ and
$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3.$
Then $\dfrac{dV}{dt} = \dfrac{1}{4}\pi h^2 \dfrac{dh}{dt}$. So $\dfrac{dh}{dt} = \dfrac{4}{\pi h^2}\dfrac{dV}{dt} = \dfrac{4}{\pi(4)^2}2\pi = \dfrac{1}{2}$.

12. (10 points) Find the perimeter of the rectangle in the first quadrant with the smallest perimeter having one edge on the x-axis and one on the y-axis and the opposite vertex on the curve $y = \dfrac{4}{x^2}$.



Solution: $P = 2x + 2y = 2x + \dfrac{8}{x^2}$ $\qquad P' = 2 - \dfrac{16}{x^3} = \dfrac{2x^3 - 16}{x^3} = 0$ at $x^3 = 8$ or $x = 2.$
So $y = \dfrac{4}{x^2} = \dfrac{4}{2^2} = 1$ and $P = 2x + 2y = 2(2) + 2(1) = 6$
$P'' = \dfrac{48}{x^4} > 0$ So $x = 2$ is a minimum.

13. (10 points) If $f(x) = \int_{x^2}^{x^3} \frac{1}{t^3+1}\,dt$, find $f'(1)$.

Solution: Let $F(t)$ be an antiderivative of $\frac{1}{t^3+1}$. In other words, $F'(t) = \frac{1}{t^3+1}$.

Then $f(x) = \int_{x^2}^{x^3} \frac{1}{t^3+1}\,dt = F(x^3) - F(x^2)$. So by the chain rule,

$$f'(x) = F'(x^3)3x^2 - F'(x^2)2x = \frac{1}{(x^3)^3+1}3x^2 - \frac{1}{(x^2)^3+1}2x$$

$$f'(1) = \frac{1}{1+1}3 - \frac{1}{1+1}2 = \frac{1}{2}$$

14. (15 points) Compute each integral.

a. $\int x^3 \cos(x^4)\,dx$

Solution: $u = x^4$ $\quad du = 4x^3\,dx$ $\quad \frac{1}{4}du = x^3\,dx$

$$\int x^3 \cos(x^4)\,dx = \frac{1}{4}\int \cos u\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4) + C$$

b. $\int \frac{x^2+1}{x^3+3x}\,dx$

Solution: $u = x^3 + 3x$ $\quad du = (3x^2+3)\,dx = 3(x^2+1)\,dx$ $\quad \frac{1}{3}du = (x^2+1)\,dx$

$$\int \frac{x^2+1}{x^3+3x}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+3x| + C$$

c. $\int_0^3 2x\sqrt{16+x^2}\,dx$ Simplify to a rational number.

Solution: $u = 16 + x^2$ $\quad du = 2x\,dx$ $\quad x = 0$ @ $u = 16$ $\quad x = 3$ @ $u = 25$

$$\int_0^3 2x\sqrt{16+x^2}\,dx = \int_{16}^{25} \sqrt{u}\,du = \frac{2u^{3/2}}{3}\Bigg|_{16}^{25} = \frac{2}{3}(25^{3/2} - 16^{3/2}) = \frac{2}{3}(125-64) = \frac{122}{3}$$