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MATH 172 Exam 3 Spring 2018

Sections 501/502 (circle one) Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Which of the following polar coordinates is NOT the point $\left(-3, \frac{\pi}{3}\right)$?

a. $(r,\theta) = \left(-3, \frac{-5\pi}{3}\right)$

b. $(r,\theta) = \left(3, \frac{2\pi}{3}\right)$ correct choice

c. $(r,\theta) = \left(-3, \frac{7\pi}{3}\right)$

d. $(r,\theta) = \left(3, \frac{-2\pi}{3}\right)$

e. $(r,\theta) = \left(3, \frac{4\pi}{3}\right)$

Solution: First, $\frac{\pi}{3}$ is the same as $\frac{\pi}{3} \pm 2\pi \equiv \frac{7\pi}{3}, \frac{-5\pi}{3}$ which gives (a) and (c).

Second, the angle opposite to $\frac{\pi}{3}$ is $\frac{\pi}{3} \pm \pi = \frac{4\pi}{3}, \frac{-2\pi}{3}$ which gives (d) and (e).

2. Find the arc length of the piece of the cardioid $r = 2 - 2\cos\theta$ in the upper half-plane.

a. 1

b. 2

c. 4

d. 8 correct choice

e. 16

Solution: $r = 2 - 2\cos\theta \qquad \frac{dr}{d\theta} = 2\sin\theta$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (2-2\cos\theta)^2 + (2\sin\theta)^2 = 4 - 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta = 8 - 8\cos\theta$$

Since $\sin^2 A = \frac{1-\cos 2A}{2}$, we have

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 16\frac{1-\cos\theta}{2} = 16\sin^2\frac{\theta}{2}$$

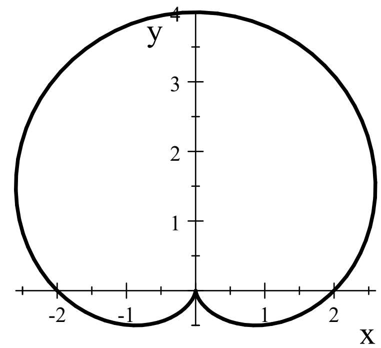
Since the upper half-plane is $0 < \theta < \pi$, the arc length is

$$L = \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^\pi \sqrt{16\sin^2\frac{\theta}{2}}\, d\theta = \int_0^\pi 4\sin\frac{\theta}{2}\, d\theta$$

$$= \left[-8\cos\frac{\theta}{2}\right]_0^\pi = -8\cos\frac{\pi}{2} + 8\cos 0 = 8$$

1-11	/55	13	/20
12	/10	14	/20
		Total	/105

3. Find the area inside the cardioid $r = 2 + 2\sin\theta$.



a. $A = \pi$

b. $A = 2\pi$

c. $A = 3\pi$

d. $A = 4\pi$

e. $A = 6\pi$ correct choice

Solution: The area is:

$$A = \frac{1}{2}\int_0^{2\pi} r^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(2 + 2\sin\theta)^2\,d\theta = \int_0^{2\pi}(2 + 4\sin\theta + 2\sin^2\theta)\,d\theta = \int_0^{2\pi}(3 + 4\sin\theta - \cos 2\theta)\,d\theta$$

$$= \left[3\theta - 4\cos\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = (6\pi - 4) - (-4) = 6\pi$$

4. $\lim\limits_{n\to\infty}\dfrac{n^2 + n}{2n^2 + (-1)^n} =$

a. 1

b. 2

c. $\frac{1}{2}$ correct choice

d. $\frac{1}{3}$

e. $\frac{2}{3}$

Solution: $\lim\limits_{n\to\infty}\dfrac{n^2 + n}{2n^2 + (-1)^n}\cdot\dfrac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim\limits_{n\to\infty}\dfrac{1 + \frac{1}{n}}{2 + \frac{(-1)^n}{n^2}} = \dfrac{1}{2}$

5. $\lim\limits_{n\to\infty}\left(\sqrt{n^2 - 4n} - n\right)$

a. -2 correct choice

b. 0

c. 2

d. 4

e. ∞

Solution: $\lim\limits_{n\to\infty}\left(\sqrt{n^2 - 4n} - n\right)\cdot\dfrac{\sqrt{n^2 - 4n} + n}{\sqrt{n^2 - 4n} + n} = \lim\limits_{n\to\infty}\dfrac{n^2 - 4n - n^2}{\sqrt{n^2 - 4n} + n} = \lim\limits_{n\to\infty}\dfrac{-4n}{\sqrt{n^2 - 4n} + n}$

$$= \lim_{n\to\infty}\frac{-4n}{\sqrt{n^2 - 4n} + n}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{-4}{\sqrt{1 - \frac{4}{n}} + 1} = \frac{-4}{2} = -2$$

6. $\lim_{n\to\infty} n^{(3/\ln n)} =$

- **a.** 0
- **b.** e
- **c.** e^3 correct choice
- **d.** 1
- **e.** 3

Solution: $\lim_{n\to\infty} n^{(3/\ln n)} = \lim_{n\to\infty} (e^{\ln n})^{(3/\ln n)} = \lim_{n\to\infty} e^{\frac{3\ln n}{\ln n}} = e^3$

7. $\sum_{n=1}^{\infty} \frac{3^{2n}}{2^{4n+1}} =$

- **a.** $\frac{3}{7}$
- **b.** $\frac{3}{14}$
- **c.** $\frac{9}{7}$
- **d.** $\frac{9}{14}$ correct choice
- **e.** diverges

Solution: Geometric:

$$a_n = \frac{3^{2n}}{2^{4n+1}} = \frac{1}{2}\left(\frac{9}{16}\right)^n \qquad a = a_1 = \frac{1}{2}\frac{9}{16} = \frac{9}{32} \qquad r = \frac{9}{16} < 1$$

$$\sum_{n=1}^{\infty} \frac{3^{2n}}{2^{4n+1}} = \frac{a}{1-r} = \frac{\frac{9}{32}}{1-\frac{9}{16}} = \frac{9}{32-18} = \frac{9}{14}$$

8. $\sum_{n=1}^{\infty} \left(\frac{n-1}{n} - \frac{n}{n+1}\right) =$

- **a.** -1 correct choice
- **b.** 0
- **c.** 1
- **d.** 2
- **e.** diverges

Solution: Telescoping:

$$S_k = \sum_{n=1}^{k} \left(\frac{n-1}{n} - \frac{n}{n+1}\right) = \left(\frac{0}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{2}{3}\right) + \cdots + \left(\frac{k-1}{k} - \frac{k}{k+1}\right) = 0 - \frac{k}{k+1}$$

$$\sum_{n=1}^{\infty} \left(\frac{n-1}{n} - \frac{n}{n+1}\right) = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(-\frac{k}{k+1}\right) = -1$$

9. $\sum_{n=1}^{\infty} \frac{n}{n^2+1} =$

a. -1
b. 0
c. 1
d. 2
e. diverges   correct choice

Solution: Integral Test: $\frac{n}{n^2+1}$ is a continuous decreasing function of n.
$\int_1^{\infty} \frac{n}{n^2+1}\,dn = \left[\frac{1}{2}\ln(n^2+1)\right]_1^{\infty} = \infty$ So the series also diverges.

10. $\sum_{n=1}^{\infty} \frac{n-1}{n}$

a. -1
b. 0
c. 1
d. 2
e. diverges   correct choice

Solution: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n-1}{n} = 1 \neq 0$ So the series diverges by the n^{th} Term Divergence Test.

11. Given a sequence a_n, suppose the partial sum is $S_k = \sum_{n=2}^{k} a_n = \ln\left(\frac{k}{k+1}\right)$. What is a_n?

HINT: What is S_{k-1}? What is the relation between S_k and S_{k-1}?

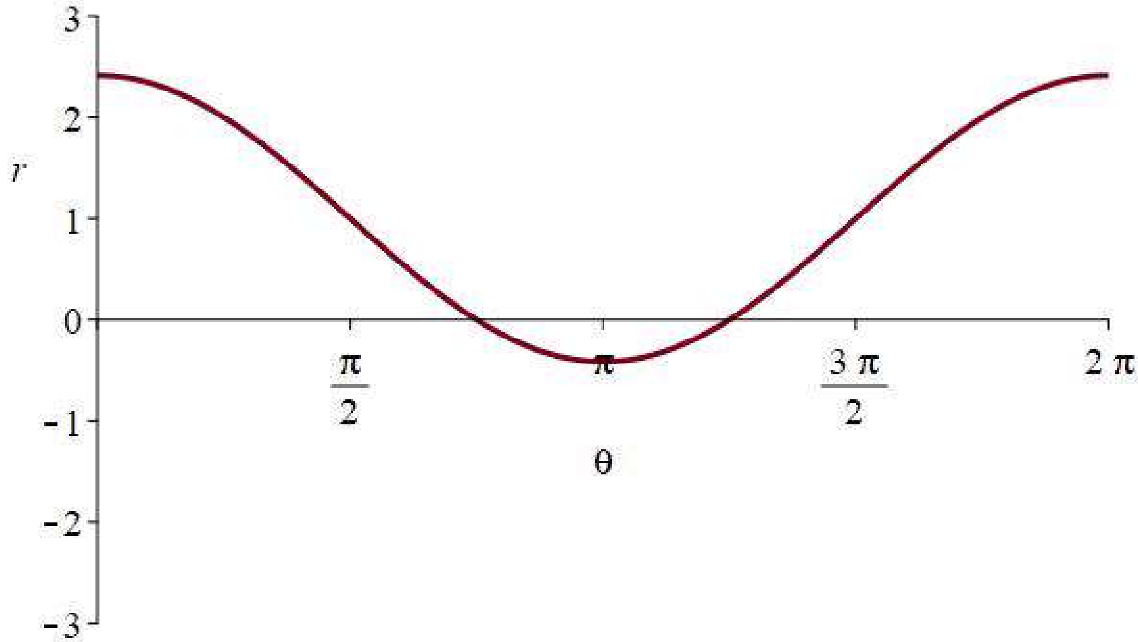
a. $a_n = \ln\left(\frac{n^2}{n^2-1}\right)$ correct choice
b. $a_n = \ln\left(\frac{n^2}{n^2+1}\right)$
c. $a_n = \ln\left(\frac{n^2-n}{n^2+n}\right)$
d. $a_n = \ln\left(\frac{n^2+n}{n^2-n}\right)$
e. Insufficient information to determine a_n.

Solution: $S_{k-1} = \ln\left(\frac{k-1}{k}\right)$ $a_k = S_k - S_{k-1} = \ln\left(\frac{k}{k+1}\right) - \ln\left(\frac{k-1}{k}\right) = \ln\left(\frac{k^2}{k^2-1}\right)$
Replace k by n: $a_n = \ln\left(\frac{n^2}{n^2-1}\right)$

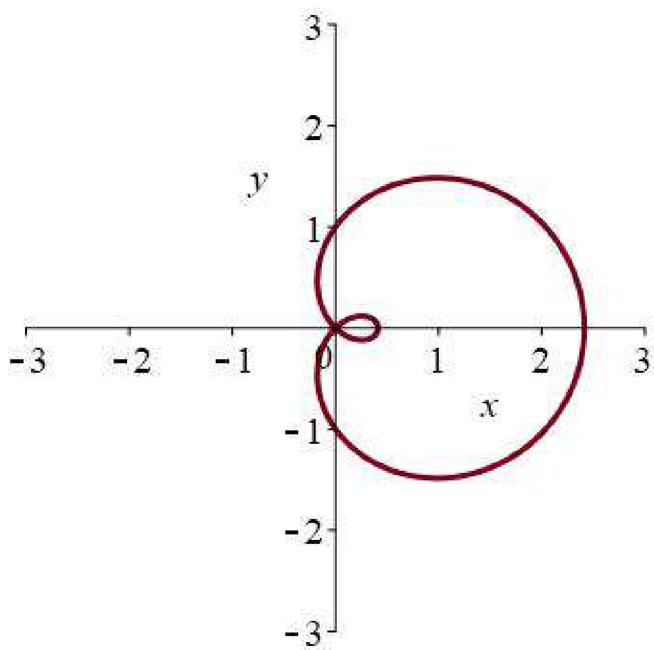
Work Out: (Points indicated. Part credit possible. Show all work.)

12. $(10$ points$)$ Consider the polar curve $r = 1 + \sqrt{2}\cos\theta$. Approximate $\sqrt{2} \approx 1.4$.

a. Plot a rectangular graph of r as a function of θ with r vertical and θ horizontal.
(You don't need to be too precise, but be careful with the values at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.)



b. Plot a polar graph of r as a function of θ with r radial and θ angular.
(You don't need to be too precise, but be careful with the values at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.)



13. $(20$ points$)$ Use a Comparison Theorem to determine whether each of the following series converges or diverges. Clearly state the comparison series, why the comparison series converges or diverges and why the original series converges or diverges. For each conclusion, name the Convergence Test.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n^2}$

Solution: For large n, we know $n^2 \gg n$. So we take the comparison series to be $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

This comparison series converges because it is a p-series with $p > 1$.

Further, $n + n^2 > n^2$. So $\dfrac{1}{n+n^2} < \dfrac{1}{n^2}$. So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n^2}$ also converges by the Simple Comparison Test.

b. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

Solution: For large n, we know $n \gg \sqrt{n}$. So we take the comparison series to be $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.

This comparison series diverges because it is harmonic (a p-series with $p = 1$).

Further, $n + \sqrt{n} > n$. So $\dfrac{1}{n+\sqrt{n}} < \dfrac{1}{n}$. So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}} < \sum_{n=1}^{\infty} \frac{1}{n}$.

This inequality is backwards for the Simple Comparison Test. So we need to do something else.

METHOD 1: Since $n > \sqrt{n}$, we have $2n > n + \sqrt{n}$. So $\dfrac{1}{2n} < \dfrac{1}{n+\sqrt{n}}$.

So $\displaystyle\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} < \sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+n^2}$ also diverges by the Simple Comparison Test.

METHOD 2: We compute the limit

$$L = \lim_{n\to\infty} \frac{\dfrac{1}{n+\sqrt{n}}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = 1$$

Since $0 < L = 1 < \infty$, we conclude $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$ behave the same by the Limit Comparison Test. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$.

14. (20 points) Consider the recursively defined sequence

$$a_1 = 3 \qquad a_{n+1} = \sqrt{6a_n - 8}$$

a. Write out the first 3 terms of the sequence:

Solution: $a_1 = \underline{3}$ $\quad a_2 = \underline{\sqrt{18-8} = \sqrt{10}}$ $\quad a_3 = \underline{\sqrt{6\sqrt{10} - 8}}$

b. Use mathematical induction to prove the sequence is increasing.
(Be sure to state the formula you need to prove.)

Solution: We need to prove: $a_n < a_{n+1}$.

Initialization: $3 < \sqrt{10}$. So $a_1 < a_2$.

Induction: Assume $a_{k-1} < a_k$.

Then $6a_{k-1} < 6a_k$, $\quad 6a_{k-1} - 8 < 6a_k - 8$ and $\sqrt{6a_{k-1} - 8} < \sqrt{6a_{k-1} - 8}$ or $a_k < a_{k+1}$.

c. Use mathematical induction to prove the sequence is bounded above by 6.
(Be sure to state the formula you need to prove.)

Solution: We need to prove: $a_n < 6$.

Initialization: $3 < \sqrt{10} < 6$. So $a_1 < a_2 < 6$.

Induction: Assume $a_{k-1} < 6$.

Then $6a_{k-1} < 36$, $\quad 6a_{k-1} - 8 < 28$ and $\sqrt{6a_{k-1} - 8} < \sqrt{28} < \sqrt{36} = 6$ or $a_k < 6$.

d. State a theorem which guarantees the sequence converges.

Solution: Bounded Monotonic Sequence Theorem:

An increasing sequence, which is bounded above, converges.

A bounded, monotonic sequence converges

e. Find the limit of the sequence.

Solution: Let $L = \lim\limits_{n\to\infty} a_n$. Then $L = \lim\limits_{n\to\infty} a_{n+1}$ also. Take the limit of the recursion formula:

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6a_n - 8} = \sqrt{6\lim_{n\to\infty} a_n - 8}$$

So:

$$L = \sqrt{6L - 8} \quad \text{or} \quad L^2 = 6L - 8 \quad \text{or} \quad L^2 - 6L + 8 = 0$$

$$\text{or} \quad (L-2)(L-4) = 0 \quad \text{or} \quad L = 2, 4.$$

Since the sequence starts at $a_1 = 3$ and increases, we must have $\lim\limits_{n\to\infty} a_n = 4$.