Name________________________________

MATH 172 Final Exam Spring 2019

Sections 501 Solutions P. Yasskin

15 Multiple Choice: (4 points each. No part credit.)

1. Compute $\int 3x^2 \ln x\, dx$.

a. $6x\ln x - 6x + C$

b. $x^3 \ln x - \dfrac{x^3}{3} + C$ correct choice

c. $6x\ln x + 6x + C$

d. $x^3 \ln x + \dfrac{x^3}{3} + C$

e. $\dfrac{x^3}{3}\ln x - \dfrac{x^3}{9} + C$

Solution: Integration by Parts: $u = \ln x \quad dv = 3x^2\,dx \quad du = \frac{1}{x}\,dx \quad v = x^3$

$$\int 3x^2 \ln x\, dx = x^3 \ln x - \int \frac{x^3}{x}\,dx = x^3\ln x - \frac{x^3}{3} + C$$

2. Compute $\int \sec^4\theta\, d\theta$.

a. $\dfrac{(\ln|\sec\theta + \tan\theta|)^5}{5} + C$

b. $\dfrac{\tan^5\theta}{5} - \dfrac{2\tan^3\theta}{3} + \tan\theta + C$

c. $\dfrac{\tan^5\theta}{5} + \dfrac{2\tan^3\theta}{3} + \tan\theta + C$

d. $\dfrac{\tan^3\theta}{3} - \tan\theta + C$

e. $\dfrac{\tan^3\theta}{3} + \tan\theta + C$ correct choice

Solution: Substitute $u = \tan\theta, \quad du = \sec^2\theta\,d\theta$:

$$\int \sec^4\theta\,d\theta = \int(\tan^2\theta + 1)\sec^2\theta\,d\theta = \int(u^2+1)\,du = \frac{u^3}{3} + u = \frac{\tan^3\theta}{3} + \tan\theta + C$$

1-15	/60	17	/15
16	/15	18	/15
		Total	/105

3. Compute $\int \sqrt{4-x^2}\,dx$.

a. $\arcsin\frac{x}{2} + \frac{x}{3}(4-x^2)^{3/2} + C$

b. $2\arcsin\frac{x}{2} - x\sqrt{4-x^2} + C$

c. $2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + C$ correct choice

d. $\arcsin\frac{x}{2} - x\sqrt{4-x^2} + C$

e. $\arcsin\frac{x}{2} + x(4-x^2)^{3/2} + C$

Solution: $x = 2\sin\theta \qquad dx = 2\cos\theta\,d\theta$

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4-4\sin^2\theta}\,2\cos\theta\,d\theta = 4\int \cos^2\theta\,d\theta = 2\int(1+\cos 2\theta)\,d\theta = 2\theta + \sin 2\theta + C$$

$$= 2\theta + 2\sin\theta\cos\theta + C = 2\arcsin\frac{x}{2} + 2\frac{x}{2}\frac{\sqrt{4-x^2}}{2} + C = 2\arcsin\frac{x}{2} + \frac{x}{2}\sqrt{4-x^2} + C$$

4. The integral $\int_1^\infty \frac{1}{x^3+\sqrt[3]{x}}\,dx$.

a. converges by comparison to $\int_1^\infty \frac{1}{x^3}\,dx$. correct choice

b. diverges by comparison to $\int_1^\infty \frac{1}{x^3}\,dx$.

c. converges by comparison to $\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx$.

d. diverges by comparison to $\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx$.

Solution: x is larger than $\sqrt[3]{x}$ for large x. So we compare to $\int_1^\infty \frac{1}{x^3}\,dx$ which converges because:

$$\int_1^\infty \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_1^\infty = 0 - -\frac{1}{2} = \frac{1}{2} \quad \text{which is finite.}$$

Since $\frac{1}{x^3+\sqrt[3]{x}} < \frac{1}{x^3}$, (There's more in the bottom.) the integral $\int_1^\infty \frac{1}{x^3+\sqrt[3]{x}}\,dx$ also converges.

5. Find the average value of the function $f(x) = \frac{1}{1+x^2}$ on the interval $\left[0, \sqrt{3}\right]$.

a. $\frac{\ln 4}{\sqrt{3}}$

b. $\frac{\ln 4}{2\sqrt{3}}$

c. $\frac{\pi}{6\sqrt{3}}$

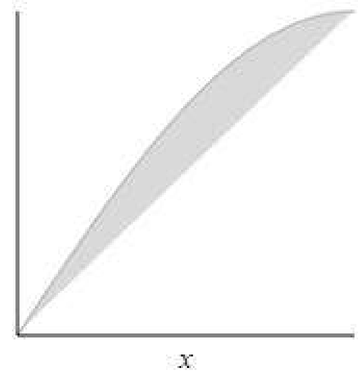
d. $\frac{\pi}{3\sqrt{3}}$ correct choice

e. $\frac{\pi}{2\sqrt{3}}$

Solution: $\int_0^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \left[\arctan x\right]_0^{\sqrt{3}} = \arctan\sqrt{3} = \frac{\pi}{3}$ since $\arctan\sqrt{3} = \frac{\pi}{3}$ and $\arctan 0 = 0$.

So $f_{\text{ave}} = \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} \frac{1}{1+x^2}\,dx = \frac{\pi}{3\sqrt{3}}$

6. The region between $y = \sin x$ and $y = \frac{2x}{\pi}$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the y-axis. Which integral gives the volume swept out?



a. $V = \int_0^{\pi/2} 2\pi x\left(\frac{2x}{\pi} - \sin x\right) dx$

b. $V = \int_0^{\pi/2} 2\pi\left(\sin^2 x - \frac{4x^2}{\pi^2}\right) dx$

c. $V = \int_0^{\pi/2} 2\pi x\left(\sin x - \frac{2x}{\pi}\right) dx$ correct choice

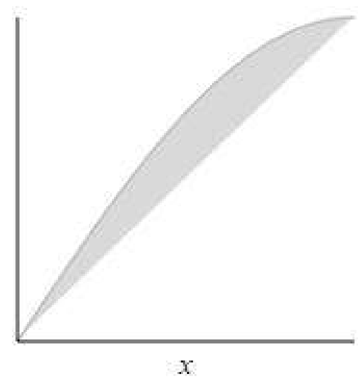
d. $V = \int_0^{\pi/2} \pi\left(\frac{4x^2}{\pi^2} - \sin^2 x\right) dx$

e. $V = \int_0^{\pi/2} \pi\left(\sin^2 x - \frac{4x^2}{\pi^2}\right) dx$

Solution: Do an x-integral. Slices are vertical. They rotate about the y-axis into cylinders.

$$V = \int_0^{\pi/2} 2\pi r h\, dx = \int_0^{\pi/2} 2\pi x\left(\sin x - \frac{2x}{\pi}\right) dx$$

7. The region between $y = \sin x$ and $y = \frac{2x}{\pi}$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the x-axis. Which integral gives the volume swept out?



a. $V = \int_0^{\pi/2} 2\pi x\left(\frac{2x}{\pi} - \sin x\right) dx$

b. $V = \int_0^{\pi/2} 2\pi\left(\sin^2 x - \frac{4x^2}{\pi^2}\right) dx$

c. $V = \int_0^{\pi/2} 2\pi x\left(\sin x - \frac{2x}{\pi}\right) dx$

d. $V = \int_0^{\pi/2} \pi\left(\frac{4x^2}{\pi^2} - \sin^2 x\right) dx$

e. $V = \int_0^{\pi/2} \pi\left(\sin^2 x - \frac{4x^2}{\pi^2}\right) dx$ correct choice

Solution: Do an x-integral. Slices are vertical. They rotate about the x-axis into washers.

$$V = \int_0^{\pi/2} \pi(R^2 - r^2)\, dx = \int_0^{\pi/2} \pi\left(\sin^2 x - \frac{4x^2}{\pi^2}\right) dx$$

8. Find the area inside the first loop of the spiral $r = \theta$ for $0 \le \theta \le 2\pi$.

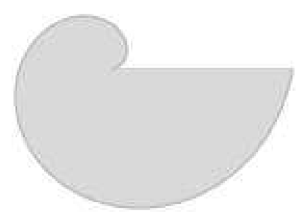
a. $2\pi^2$

b. $\frac{4\pi^3}{3}$ correct choice

c. $\frac{4\pi^2}{3}$

d. $\frac{2\pi^2}{3}$

e. $\frac{8\pi^3}{3}$



Solution: $A = \int_0^{2\pi} \frac{1}{2} r^2\, d\theta = \int_0^{2\pi} \frac{1}{2}\theta^2\, d\theta = \left[\frac{\theta^3}{6}\right]_0^{2\pi} = \frac{8\pi^3}{6} = \frac{4\pi^3}{3}$

9. Find the center of mass of a bar which is 6 cm long and has density $\delta = x + x^2$ where x is measured from one end.

a. $\frac{22}{5}$ correct choice

b. $\frac{5}{22}$

c. $\frac{11}{5}$

d. $\frac{5}{11}$

e. $\frac{8}{5}$

Solution: $M = \int_0^6 \delta\,dx = \int_0^6 (x + x^2)\,dx = \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^6 = \frac{36}{2} + \frac{216}{3} = 18 + 72 = 90$

$M_1 = \int_0^6 x\delta\,dx = \int_0^6 (x^2 + x^3)\,dx = \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^6 = \frac{216}{3} + \frac{1296}{4} = 72 + 324 = 396$

$\bar{x} = \frac{M_1}{M} = \frac{396}{90} = \frac{22}{5}$

10. The series $\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n(n^3 + \sqrt[3]{n}\,)}$ has radius of convergence $R = 2$. Find its interval of convergence.

a. $(1,5)$

b. $[1,5)$

c. $(1,5]$

d. $[1,5]$ correct choice

Solution: At $x = 1$: $\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n(n^3 + \sqrt[3]{n}\,)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^3 + \sqrt[3]{n}}$ which converges by the Alternating Series Test.

At $x = 5$: $\sum_{n=0}^{\infty} \frac{(2)^n}{2^n(n^3 + \sqrt[3]{n}\,)} = \sum_{n=0}^{\infty} \frac{1}{n^3 + \sqrt[3]{n}}$ which converges by comparison with $\sum_{n=0}^{\infty} \frac{1}{n^3}$ which is a convergent p-series since $p = 3 > 1$.

11. Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}(x-3)^n$.

a. 0

b. 1

c. 2

d. 4

e. ∞ correct choice

Solution: Ratio Test:

$\rho = \lim_{n\to\infty} \frac{(n+1)!|x-3|^{n+1}}{(2n+2)!} \frac{(2n)!}{n!|x-3|^n} = |x-3| \lim_{n\to\infty} \frac{n+1}{(2n+2)(2n+1)} = 0 < 1$ forall x. So $R = \infty$.

12. Compute $\lim_{n\to\infty} \dfrac{(-1)^n 4n^3 + n}{(-1)^n 2n^3 + 3n}$.

a. $\frac{1}{3}$
b. $\frac{4}{5}$
c. 2 correct choice
d. ∞
e. divergent but not to $\pm\infty$

Solution: $\lim_{n\to\infty} \dfrac{(-1)^n 4n^3 + n}{(-1)^n 2n^3 + 3n} \dfrac{n^{-3}}{n^{-3}} = \lim_{n\to\infty} \dfrac{(-1)^n 4 + n^{-2}}{(-1)^n 2 + 3n^{-2}} = 2$

13. The series $\sum_{n=1}^{\infty} \dfrac{3n^2}{n^3+2}$

a. converges by Simple Comparison with $\sum_{n=1}^{\infty} \dfrac{3}{n}$.
b. diverges by Simple Comparison with $\sum_{n=1}^{\infty} \dfrac{3}{n}$.
c. converges by the Integral Test.
d. diverges by the Integral Test. correct choice
e. diverges by the n^{th} Term Divergence Test.

Solution: The series $\sum_{n=1}^{\infty} \dfrac{3}{n}$ diverges because it is harmonic. But $\dfrac{3n^2}{n^3+2} < \dfrac{3}{n}$, so Simple Comparison fails.

$\int_1^{\infty} \dfrac{3n^2}{n^3+2}\, dn = \Big[\ln(n^3+2) \Big]_1^{\infty} = \infty$ So $\sum_{n=1}^{\infty} \dfrac{3n^2}{n^3+2}$ diverges by the Integral Test.

14. If the series $S = \sum_{n=1}^{\infty} \dfrac{2n}{(n^2+2)^2}$ is approximated by its 100^{th} partial sum $S_{100} = \sum_{n=1}^{100} \dfrac{2n}{(n^2+2)^2}$ find a bound on the error $E_{100} = \sum_{n=101}^{\infty} \dfrac{2n}{(n^2+2)^2}$.

a. $|E_{100}| < \dfrac{2 \cdot 100}{(100^2+2)^2}$
b. $|E_{100}| < \dfrac{2 \cdot 101}{(101^2+2)^2}$
c. $|E_{100}| < \dfrac{1}{99^2+2}$
d. $|E_{100}| < \dfrac{1}{100^2+2}$ correct choice
e. $|E_{100}| < \dfrac{1}{101^2+2}$

Solution: $|E_{100}| < \int_{100}^{\infty} \dfrac{2n}{(n^2+2)^2}\, dn = \left| \dfrac{-1}{n^2+2} \right|_{100}^{\infty} = 0 - \dfrac{-1}{100^2+2} = \dfrac{1}{100^2+2}$

Note: The series is not alternating. So we cannot use the next term (b).

15. Compute $\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9}$.

a. $-\frac{1}{3}$

b. $-\frac{1}{6}$ correct choice

c. 0

d. $\frac{1}{6}$

e. ∞

Solution: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \sin x^3 = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots$

$$\lim_{x\to 0} \frac{\sin(x^3) - x^3}{x^9} = \lim_{x\to 0} \frac{\left(x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \cdots\right) - x^3}{x^9} = -\frac{1}{3!} = -\frac{1}{6}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

16. (15 points) Compute $\int \frac{2}{x^3 - x}\, dx$.

a. Find the general partial fraction expansion. (Do not find the coefficients.)

Solution: $\frac{2}{x^3 - x} = \frac{2}{x(x-1)(x+1)} = \underline{\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}}$

b. Find the coefficients and plug them back into the expansion.

Solution: Clear the denominator: $2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

Plug in $x = 0$: $2 = A(-1)$ $A = -2$

Plug in $x = 1$: $2 = B(2)$ $B = 1$

Plug in $x = -1$: $2 = C(2)$ $C = 1$

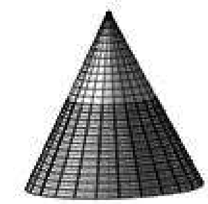
$$\frac{2}{x^3 - x} = \underline{-\frac{2}{x} + \frac{1}{x-1} + \frac{1}{x+1}}$$

c. Compute the integral.

Solution: $\int \frac{2}{x^3 - x}\, dx = \int -\frac{2}{x}\, dx + \int \frac{1}{x-1}\, dx + \int \frac{1}{x+1}\, dx = \underline{-2\ln|x| + \ln|x-1| + \ln|x+1| + C}$

17. (15 points) A water tank has the shape of a cone with the vertex at the top. Its height is $H = 20$ ft and its radius is $R = 10$ ft. It is filled with salt water to a depth of 10 ft which weighs $\delta = 64 \frac{\text{lb}}{\text{ft}^3}$.

Find the work done to pump the water out the top of the tank.



Solution: Put the y-axis measuring down from the top.

The slice which is a distance y down from the top is a circle of radius r.

By similar triangles, $\frac{r}{y} = \frac{R}{H} = \frac{10}{20} = \frac{1}{2}$. So $r = \frac{1}{2}y$.

The area is $A = \pi r^2 = \frac{\pi y^2}{4}$ and the volume of the slice of thickness dy is $dV = A\,dy = \frac{\pi y^2}{4}\,dy$.

It weighs $dF = \delta\,dV = 64\frac{\pi y^2}{4}\,dy = 16\pi y^2\,dy$. It is lifted a distance $D = y$.

There is water between $y = 10$ and $y = 20$. So the work done is

$$W = \int_{10}^{20} D\,dF = \int_{10}^{20} y\,16\pi y^2\,dy = \left[16\pi\frac{y^4}{4}\right]_{10}^{20} = 4\pi(20^4 - 10^4) = 600\,000\pi \text{ ft-lb}$$

18. (15 points) Consider the function $f(x) = \frac{1}{x^2}$.

a. Find the 3rd degree Taylor polynomial for $f(x)$ centered at $x = 2$ by taking derivatives.

Solution: $f(x) = \frac{1}{x^2}$ $\quad f'(x) = \frac{-2}{x^3}$ $\quad f''(x) = \frac{3!}{x^4}$ $\quad f'''(x) = \frac{-4!}{x^5}$

$f(2) = \frac{1}{2^2}$ $\quad f'(2) = \frac{-2}{2^3}$ $\quad f''(2) = \frac{3!}{2^4}$ $\quad f'''(2) = \frac{-4!}{2^5}$

$$\begin{aligned} T_3 f &= f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 \\ &= \frac{1}{2^2} - \frac{2}{2^3}(x-2) + \frac{3!}{2!2^4}(x-2)^2 - \frac{4!}{3!2^5}(x-2)^3 \\ &= \frac{1}{2^2} - \frac{2}{2^3}(x-2) + \frac{3}{2^4}(x-2)^2 - \frac{4}{2^5}(x-2)^3 \end{aligned}$$

b. Find the general term of its Taylor series and write the series in summation notation.

Solution: $f^{(n)}(x) = \frac{(-1)^n(n+1)!}{x^{n+2}}$ $\quad f^{(n)}(2) = \frac{(-1)^n(n+1)!}{2^{n+2}}$

$$Tf = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{(n+1)!}{2^{n+2}}(x-2)^n = \sum_{n=0}^{\infty}(-1)^n\frac{(n+1)}{2^{n+2}}(x-2)^n$$

c. Find the radius of convergence.

Solution: $\rho = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{(n+2)|x-2|^{n+1}}{2^{n+3}}\frac{2^{n+2}}{(n+1)|x-2|^n} = \frac{|x-2|}{2}\lim_{n\to\infty}\frac{n+2}{n+1} = \frac{|x-2|}{2} < 1$

$|x-2| < 2 \qquad R = 2$