Name________________________________

MATH 172 Exam 3 Spring 2020

Sections 501 Solutions P. Yasskin

Multiple Choice: (Points indicated. No part credit.)

1. (1 points) An Aggie does not lie, cheat or steal or tolerate those who do.

True [X] False [ ]

2. (1 points) Each answer is one of the following or a sum of these:

a rational number in lowest terms, e.g. $-\frac{217}{5}$ which is entered as "-217/5"

a rational number in lowest terms times π, e.g. $\frac{217}{5}\pi$ which is entered as "217/5pi"

exponentials such as e^4 or $3^{12/5}$ which are entered as "e^4" or "3^(12/5)"

positive infinity, ∞, which is entered as "infinity"

negative infinity, $-\infty$, which is entered as "-infinity"

convergent, which is entered as "convergent"

divergent, which is entered as "divergent"

Do not leave any spaces. Do not use decimals.

I read this.

True [X] False [ ]

3. (4 points) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{4}{2^n}$.

a. 0

b. 1

c. 2

d. 4 correct choice

e. ∞

Solution: Geometric $a = \frac{4}{2} = 2$ $r = \frac{1}{2}$ $S = \frac{a}{1-r} = \frac{2}{1-\frac{1}{2}} = 4$

1-9	/30	13	/18
10-11	/28	14	/8
12	/18	Total	/102

4. (4 points) Compute $\sum_{n=1}^{\infty}\left(\frac{n}{2n-1} - \frac{n+1}{2n+1}\right)$.

a. 0

b. $\frac{1}{2}$ correct choice

c. 1

d. $\frac{3}{2}$

e. ∞

Solution: Telescoping

$$S_k = \sum_{n=1}^{\infty}\left(\frac{n}{2n-1} - \frac{n+1}{2n+1}\right) = \left(1 - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{5}\right) + \cdots + \left(\frac{k}{2k-1} - \frac{k+1}{2k+1}\right) = 1 - \frac{k+1}{2k+1}$$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(1 - \frac{k+1}{2k+1}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

5. (4 points) Compute $\lim_{n\to\infty}\left(\sqrt{n^6 + 5n^3} - \sqrt{n^6 - 4n^3}\right)$.

a. $-\infty$

b. -4

c. 9

d. $\frac{9}{2}$ correct choice

e. ∞

Solution: $\lim_{n\to\infty}\left(\sqrt{n^6 + 5n^3} - \sqrt{n^6 - 4n^3}\right)\frac{\sqrt{n^6 + 5n^3} + \sqrt{n^6 - 4n^3}}{\sqrt{n^6 + 5n^3} + \sqrt{n^6 - 4n^3}} = \lim_{n\to\infty}\frac{(n^6 + 5n^3) - (n^6 - 4n^3)}{\sqrt{n^6 + 5n^3} + \sqrt{n^6 - 4n^3}}$

$$= \lim_{n\to\infty}\frac{9n^3}{\sqrt{n^6 + 5n^3} + \sqrt{n^6 - 4n^3}} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n3}} = \lim_{n\to\infty}\frac{9}{\sqrt{1 + 5n^{-3}} + \sqrt{1 - 2n^{-3}}} = \frac{9}{2}$$

6. (4 points) Compute $\lim_{n\to\infty}\left(1 + \frac{1}{2n}\right)^{4n}$. If divergent, enter "infinity" or "-infinity".

a. e

b. e^2 correct choice

c. e^4

d. e^8

e. ∞

Solution: $\lim_{n\to\infty}\left(1 + \frac{1}{2n}\right)^{4n} = e^{\ln \lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{4n}} = \exp \lim_{n\to\infty} 4n\ln\left(1 + \frac{1}{2n}\right)$

$$= \exp \lim_{n\to\infty}\frac{4\ln\left(1 + \frac{1}{2n}\right)}{\frac{1}{n}} \overset{l'H}{=} \exp \lim_{n\to\infty}\frac{\frac{4}{1+\frac{1}{2n}}\left(\frac{-1}{2n^2}\right)}{\frac{-1}{n^2}} = \exp \lim_{n\to\infty}\frac{2}{1 + \frac{1}{2n}} = e^2$$

7. (4 points) Compute $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$. If divergent, enter "infinity" or "-infinity".

a. e^3

b. e^{-3}

c. $-e^3 - 1$

d. $e^{-3} - 1$ correct choice

e. ∞

Solution: A standard Maclaurin series is $e^x = \sum_{n=0}^{\infty} \frac{(x)^n}{n!}$.

At $x = -3$ this says $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!} = e^{-3}$. Our series starts at $n = 1$. So

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} - 1 = e^{-3} - 1$$

8. (4 points) If $S = \sum_{n=1}^{\infty} a_n$ and $S_k = \frac{k}{2k+1}$, then

a. $S = 0$

b. $S = 1$

c. $S = \frac{1}{2}$ correct choice

d. $S = \frac{1}{3}$

e. $S = \infty$

Solution: $S = \lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{k}{2k+1} = \frac{1}{2}$

9. (4 points) If the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ is approximated by the 99^{th} partial sum $S_{99} = \sum_{n=1}^{99} \frac{(-1)^{n+1}}{n^3} \approx 0.90154318486844623867$, how many digits of accuracy are guaranteed in this approximation? For example, if the error is $|E_{99}| < 10^{-5}$, then only the digits 0.9015 are accurate, and you would answer 4.

a. 4

b. 5 correct choice

c. 10

d. 100

e. 1000000

Solution: Since this is an alternating, decreasing series, the error is less than the absolute value of the next term which is $|E_{99}| < \frac{1}{100^3} = 10^{-6}$. So the approximation is good to 5 terms.

10. $\left(14\text{ points}\right)$ The series $\sum_{n=2}^{\infty} \frac{1}{n-1}$ can be shown to diverge by which of the following Convergence Tests? Check Yes for all that work; check No for all that don't work.

a. n^{th}-Term test for Divergence:

[_____]Yes [__X__]No $\lim_{n\to\infty} \frac{1}{n-1} = 0$ Test Fails

b. Integral Test:

[__X__]Yes [_____]No $\int_2^{\infty} \frac{1}{n-1}\,dn = \left[\ln(n-1)\right]_2^{\infty} = \infty$

c. p-Series Test:

[__X__]Yes [_____]No $\sum_{n=2}^{\infty} \frac{1}{n-1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ p-series with $p = 1$ harmonic

d. Simple Comparison Test comparing to $\sum_{n=2}^{\infty} \frac{1}{n}$:

[__X__]Yes [_____]No $\frac{1}{n-1} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges

e. Limit Comparison Test comparing to $\sum_{n=2}^{\infty} \frac{1}{n}$:

[__X__]Yes [_____]No $\lim_{n\to\infty} \frac{n}{n-1} = 1$ and $0 < 1 < \infty$

f. Ratio Test:

[_____]Yes [__X__]No $\lim_{n\to\infty} \frac{n-1}{(n+1)-1} = 1$ Test Fails

g. Alternating Series Test:

[_____]Yes [__X__]No This series is not alternating.

11. (14 points) The series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ can be shown to converge by which of the following Convergence Tests? Check Yes for all that work; check No for all that don't work.

a. n^{th}-Term test for Divergence:

[ ] Yes [X] No $\lim_{n\to\infty} \frac{1}{n^2-1} = 0$ Test Fails

b. Integral Test:

[X] Yes [ ] No $\int_2^{\infty} \frac{1}{n^2-1}\,dn = \left[\frac{1}{2}\ln\left(\frac{n-1}{n+1}\right)\right]_2^{\infty} = \frac{1}{2}\ln 3 < \infty$

c. p-Series Test:

[ ] Yes [X] No This is not a p-series.

d. Simple Comparison Test comparing to $\sum_{n=2}^{\infty} \frac{1}{n^2}$:

[ ] Yes [X] No $\frac{1}{n^2-1} > \frac{1}{n^2}$ Wrong inequality.

e. Limit Comparison Test comparing to $\sum_{n=2}^{\infty} \frac{1}{n^2}$:

[X] Yes [ ] No $\lim_{n\to\infty} \frac{n^2}{n^2-1} = 1$ and $0 < 1 < \infty$

f. Ratio Test:

[ ] Yes [X] No $\lim_{n\to\infty} \frac{n^2-1}{(n+1)^2-1} = 1$ Test Fails

g. Alternating Series Test:

[ ] Yes [X] No This series is not alternating.

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (18 points) Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{2^n}{1+\sqrt{n}}(x-3)^n$.

a. Find the radius of convergence and state the open interval of absolute convergence.

$R =$ ____. Absolutely convergent on (_____,_____).

Solution: To find the radius, we use the Ratio Test. $|a_n| = \frac{2^n|x-3|^n}{1+\sqrt{n}}$ $|a_{n+1}| = \frac{2^{n+1}|x-3|^{n+1}}{1+\sqrt{n+1}}$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2^{n+1}|x-3|^{n+1}}{1+\sqrt{n+1}} \frac{1+\sqrt{n}}{2^n|x-3|^n} = 2|x-3| \lim_{n\to\infty} \frac{1+\sqrt{n}}{1+\sqrt{n+1}} = 2|x-3| < 1$$

$|x-3| < \frac{1}{2}$ So $R = \frac{1}{2}$. Absolutely convergent on $\left(\frac{5}{2}, \frac{7}{2}\right)$

b. Check the **Left** Endpoint:

$x =$_____ The series becomes__________________________ Circle one:

Reasons: Convergent

Solution: $x = \frac{5}{2}$: $\sum_{n=1}^{\infty} \frac{2^n}{1+\sqrt{n}}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$ Divergent

This converges by the Alternating Series Test because $\frac{1}{1+\sqrt{n}}$ is positive, decreasing and $\lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$.

c. Check the **Right** Endpoint:

$x =$_____ The series becomes__________________________ Circle one:

Reasons: Convergent

Solution: $x = \frac{7}{2}$: $\sum_{n=1}^{\infty} \frac{2^n}{1+\sqrt{n}}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ Divergent

Compare this to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a p-series with $p = \frac{1}{2} < 1$ and so diverges.

We can't use the Simple Comparison Test because $\frac{1}{1+\sqrt{n}} < \frac{1}{\sqrt{n}}$. So we compute:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1+\sqrt{n}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+1} = 1.$$

Since $0 < L = 1 < \infty$, the series $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ diverges by the Limit Comparison Test.

d. State the Interval of Convergence. Interval= ____________

Solution: The Interval of Convergence.is: $\left[\frac{5}{2}, \frac{7}{2}\right)$

13. $\left(18 \text{ points}\right)$ Determine whether the recursively defined sequence $a_1 = 2\sqrt{6}$ and $a_{n+1} = \dfrac{(a_n)^2 + 16}{10}$ is convergent or divergent. If convergent, find the limit. If divergent, say infinity or -infinity.

a. Find the first 3 terms: $a_1 =$ ______ $a_2 =$ ______ $a_3 =$ ______

Solution: $a_1 =$ ___$2\sqrt{6}$___ $a_2 =$ ___4___ $a_3 =$ ___3.2___

b. Assuming the limit $\lim\limits_{n\to\infty} a_n$ exists, find the possible limits.

Solution: Assume $\lim\limits_{n\to\infty} a_n = L$. Then $\lim\limits_{n\to\infty} a_{n+1} = L$ also. From the recursion relation:

$$L = \frac{L^2 + 16}{10} \qquad L^2 - 10L + 16 = 0 \qquad (L-2)(L-8) = 0 \qquad L = 2, 8$$

c. Prove the sequence is bounded or unbounded above or below (as appropriate).

Solution: It looks like the terms are always > 0 or from the possible limits, always > 2. We will show it's bounded below by 0. So we want to prove $a_n > 0$.
Initialization Step: $a_1 = 2\sqrt{6} > 0$
Induction Step: Assume $a_k > 0$. We need to prove $a_{k+1} > 0$.
Proof:

$$a_k > 0 \quad \Rightarrow \quad (a_k)^2 > 0 \quad \Rightarrow \quad \frac{(a_k)^2 + 16}{10} > \frac{16}{10} > 0 \quad \Rightarrow \quad a_{k+1} > 0$$

d. Prove the sequence is increasing or decreasing (as appropriate).

Solution: From the first 3 terms, we expect the sequence is decreasing. So we want to prove $a_{n+1} < a_n$.
Initialization Step: $a_1 = 2\sqrt{6} > 2\sqrt{4} = 4 = a_2$
Induction Step: Assume $a_{k+1} < a_k$. We need to prove $a_{k+2} < a_{k+1}$.
Proof: We know $a_n > 0$. So:

$$a_{k+1} < a_k \quad \Rightarrow \quad (a_{k+1})^2 < (a_k)^2 \quad \Rightarrow \quad \frac{(a_{k+1})^2 + 16}{10} < \frac{(a_k)^2 + 16}{10} \quad \Rightarrow \quad a_{k+2} < a_{k+1}$$

e. State whether the sequence is convergent or divergent and name the theorem. If convergent, determine the limit. If divergent, determine if it is infinity or -infinity.

Solution: The sequence is convergent by the Bounded Monotonic Sequence Theorem. Since it has a limit and the limit must be 2 or 8 and it decreases from 4 the limit must be $\lim\limits_{n\to\infty} a_n = 2$.

14. $(8$ points$)$ A ball is dropped from a height of 72 feet. Each time it bounces it reaches a height which is $\frac{1}{2}$ of the height on the previous bounce. What is the total distance travelled by the ball (with an infinite number of bounces)?

Solution: The ball drops 72 ft, rises and falls 36 ft, rises and falls 18 ft, etc. The total distance is:

$$D = 72 + 2(36 + 18 + 9 + \cdots) = 72 + 2\sum_{n=0}^{\infty} 36\left(\frac{1}{2}\right)^n = 72 + 2\left(\frac{36}{1 - \frac{1}{2}}\right) = 72 + 2(72) = 216$$