Name________________________________

MATH 172 | Final Exam | Spring 2020

Sections 501 | Solutions | P. Yasskin

Multiple Choice: (Points indicated. No part credit.)

1. $(1$ points$)$ An Aggie does not lie, cheat or steal or tolerate those who do.

True $\boxed{\text{X}}$ False $\boxed{}$

2. $(1$ points$)$ Each answer is one of the following or a sum of these:

a rational number in lowest terms, e.g. $-\frac{217}{5}$ which is entered as "-217/5"

a rational number in lowest terms times π, e.g. $\frac{217}{5}\pi$ which is entered as "217/5pi"

square roots are entered using sqrt, e.g. $\frac{1}{2}\sqrt{3}\,\pi$ which is entered as "1/2sqrt(3)pi"

positive infinity, ∞, which is entered as "infinity"

negative infinity, $-\infty$, which is entered as "-infinity"

convergent, which is entered as "convergent"

divergent, which is entered as "divergent"

Do not leave any spaces. Do not use decimals.

I read this.

True $\boxed{\text{X}}$ False $\boxed{}$

3. $(5$ points$)$ The area below $y = x^2$ for $0 \le x \le 2$ is rotated about the y axis. Find the volume swept out.

a. 8π correct choice

b. 16π

c. $\frac{8}{3}\pi$

d. $\frac{16}{3}\pi$

e. $\frac{8}{9}\pi$

Solution: We do an x integral. The slices are vertical and rotate about the y axis to sweep out cylinders with radius $r = x$ and height $h = y = x^2$. The volume is:

$$V = \int_0^2 2\pi rh\,dx = 2\int_0^2 \pi x x^2\,dx = \left[\frac{\pi x^4}{2}\right]_0^2 = 8\pi$$

1-9	/37	11	/20
10	/25	12	/20
		Total	/102

4. (5 points) Find the center of mass of a bar which is 4 cm long and has density $\delta = 6x - x^2$ where x is measured from one end.

a. 3

b. $\frac{5}{12}$

c. $\frac{12}{5}$ correct choice

d. $\frac{80}{3}$

e. 64

Solution: $M = \int_0^4 \delta\,dx = \int_0^4 (6x - x^2)\,dx = \left[3x^2 - \frac{x^3}{3}\right]_0^4 = 48 - \frac{64}{3} = \frac{80}{3}$

$M_1 = \int_0^4 x\delta\,dx = \int_0^4 (6x^2 - x^3)\,dx = \left[2x^3 - \frac{x^4}{4}\right]_0^4 = 128 - 64 = 64$

$\bar{x} = \frac{M_1}{M} = 64 \cdot \frac{3}{80} = \frac{12}{5}$

5. (5 points) The series $\sum_{n=2}^{\infty} \frac{(-1)^2}{\sqrt{n} + n^2}$ is

a. absolutely convergent correct choice

b. conditionally convergent

c. divergent

d. conditionally divergent

Solution: The original series is an alternating, decreasing series and $\lim_{n\to\infty} \frac{1}{\sqrt{n} + n^2} = 0$.

So the series converges. The related absolute series is $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n} + n^2}$ which converges by comparison with $\sum_{n=2}^{\infty} \frac{1}{n^2}$. So the series is absolutely convergent.

6. (5 points) Compute $\lim_{x\to 0} \frac{\sin(x^5) - x^5}{x^{15}}$. If divergent, enter "infinity" or "-infinity".

a. $-\infty$

b. $-\frac{1}{6}$ correct choice

c. 0

d. $\frac{1}{6}$

e. ∞

Solution: $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \qquad \sin x^5 = x^5 - \frac{x^{15}}{6} + \frac{x^{25}}{120} - \cdots$

$$\lim_{n\to\infty} \frac{\sin(x^5) - x^5}{x^{15}} = \lim_{n\to\infty} \frac{\left(x^5 - \frac{x^{15}}{6} + \frac{x^{25}}{120} - \cdots\right) - x^5}{x^{15}} = \lim_{n\to\infty} \frac{-\frac{x^{15}}{6} + \frac{x^{25}}{120} - \cdots}{x^{15}}$$

$$= \lim_{n\to\infty}\left(-\frac{1}{6} + \frac{x^{10}}{120} + \cdots\right) = -\frac{1}{6}$$

7. (5 points) Compute $\int_0^1 x^2 e^x\,dx$.

a. $e-2$ correct choice

b. $e-1$

c. e

d. $e+1$

e. $e+2$

Solution: Parts: $u = x^2 \quad dv = e^x\,dx$ Then
$du = 2x\,dx \quad v = e^x$

$\int x^2 e^x\,dx = x^2 e^x - 2\int x e^x\,dx$ Parts: $u = x \quad dv = e^x\,dx$ Then
$du = dx \quad v = e^x$

$$\int x^2 e^x\,dx = x^2 e^x - 2\left(x e^x - \int e^x\,dx\right) = x^2 e^x - 2x e^x + 2e^x + C$$

$$\int_0^1 x^2 e^x\,dx = \left[x^2 e^x - 2x e^x + 2e^x\right]_0^1 = (e - 2e + 2e) - 2 = e - 2$$

8. (5 points) Compute $\int_0^{\pi/4} \tan^4\theta \sec^2\theta\,d\theta$

a. 125

b. 625

c. $\frac{4}{5}$

d. $\frac{1}{25}$

e. $\frac{1}{5}$ correct choice

Solution: Let $u = \tan\theta \quad du = \sec^2\theta\,d\theta$

$$\int_0^{\pi/4} \tan^4\theta \sec^2\theta\,d\theta = \int u^4\,du = \left[\frac{u^5}{5}\right] = \left[\frac{\tan^5\theta}{5}\right]_0^{\pi/4} = \frac{1}{5}$$

9. (5 points) Compute $\int_0^2 \frac{8}{(x^2+4)^{3/2}}\,dx$.

a. $-\sqrt{3}$

b. $-\sqrt{2}$

c. 0

d. $\sqrt{2}$ correct choice

e. $\sqrt{3}$

Solution: $x = 2\tan\theta \quad dx = 2\sec^2\theta\,d\theta$ upper limit: $2 = 2\tan\theta \Rightarrow \theta = \frac{\pi}{4}$

$$\int_0^2 \frac{8}{(x^2+4)^{3/2}}\,dx = \int_0^{\pi/4} \frac{8}{(4\tan^2\theta+4)^{3/2}}\,2\sec^2\theta\,d\theta = 2\int_0^{\pi/4} \frac{1}{(\sec^2\theta)^{3/2}}\,\sec^2\theta\,d\theta = 2\int_0^{\pi/4} \cos\theta\,d\theta$$

$$= \left[2\sin\theta\right]_0^{\pi/4} = \sqrt{2}$$

Work Out Problems: (Show all work on paper and upload in Part 2. Enter answers here.)

10. $(25$ points$)$ A paper soda cup is 8 cm tall, has a circular base of radius 4 cm and a circular top of 5 cm. So the sides are given by rotating the line $x = 4 + \frac{y}{8}$ about the y-axis.

a. Find the volume of the cup.

Solution: This is a y integral. Each horizontal slice rotates into a disk of radius $r = x = 4 + \frac{y}{8}$. So the volume is

$$V = \int_0^8 \pi r^2\, dy = \int_0^8 \pi \left(4 + \frac{y}{8}\right)^2 dy = \frac{8\pi}{3}\left(4 + \frac{y}{8}\right)^3 \Big|_0^8 = \frac{8\pi}{3}(5^3 - 4^3) = \frac{488}{3}\pi$$

b. Find the surface area of the sides of the cup.

Solution: The differential of arclength is

$$ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy = \sqrt{\left(\frac{1}{8}\right)^2 + 1}\, dy = \frac{\sqrt{65}}{8}\, dy$$

The radius of rotation is $r = x = 4 + \frac{y}{8}$. So the surface area is

$$A = \int_0^8 2\pi r\, ds = 2\pi \int_0^8 \left(4 + \frac{y}{8}\right) \frac{\sqrt{65}}{8}\, dy = \frac{\sqrt{65}\,\pi}{4}\left[4y + \frac{y^2}{16}\right]_0^8 = 9\sqrt{65}\,\pi$$

c. There is soda in the cup up to 7 cm. The density of the soda is δ and the acceleration of gravity is g. A 10 cm straw is put into the cup, so that the bottom of the straw is at the bottom of the cup. Set up the integral for the work done to suck the soda through the top of the straw. Be sure to explain how you got it on your paper. Do not compute the integral. The integral will have the form:

$$W = \boxed{c}\,\delta g \int_{\boxed{d}}^{\boxed{e}} \left(\boxed{f} - y\right)\left(\boxed{g} + \boxed{h}\,y\right)^{\boxed{i}} dy$$

Identify each of the quantities in boxes:

- $c =$
- $d =$
- $e =$
- $f =$
- $g =$
- $h =$
- $i =$

Solution: The soda at height y with thickness dy is a disk of radius $r = x = 4 + \frac{y}{8}$ and height dy. So its volume is: $dV = \pi r^2\, dy = \pi\left(4 + \frac{y}{8}\right)^2 dy$

and its weight is: $dF == \delta g dV = \delta g \pi \left(4 + \frac{y}{8}\right)^2 dy$

This disk of soda is lifted a distance: $D = 10 - y$

There is soda between: $y = 0$ and $y = 7$.

So the work is: $W = \int D\, dF = \delta g \pi \int_0^7 (10 - y)\left(4 + \frac{y}{8}\right)^2 dy$

Comparing to the template: $c = \pi$ $d = 0$ $e = 7$ $f = 10$ $g = 4$ $h = 1/8$ $i = 2$

11. (20 points) Find the coefficients in the partial fraction expansion

$$\frac{6x+8}{(x-2)^2(x^2+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+1}$$

Solution: Clear the denominator:

$$6x+8 = A(x-2)(x^2+1) + B(x^2+1) + (Cx+D)(x-2)^2 \qquad (1)$$

Plug in $x = 2$: $\quad 20 = B(5)$

$$B = 4 \qquad (2)$$

Plug in $x = 0$: $\quad 8 = A(-2)(1) + B(1) + D(-2)^2 = -2A + 4 + 4D \qquad 2A - 4D = -4$

$$A - 2D = -2 \qquad (3)$$

Plug in $x = 1$: $\quad 14 = A(-1)(2) + B(2) + (C+D)(-1)^2 = -2A + 8 + C + D$

$$2A - C - D = -6 \qquad (4)$$

Coeff of x^3: $\quad 0 = A + C$

$$C = -A \qquad (5)$$

Plug (5) into (4):

$$3A - D = -6 \quad \text{or} \quad D = 3A + 6 \qquad (6)$$

Plug (6) into (3):

$$A - 2(3A+6) = -2 \quad \text{or} \quad -5A - 12 = -2 \quad \text{or} \quad A = -2 \qquad (7)$$

Substitute back:

$$C = 2 \qquad D = 0$$

So

$$\frac{6x+8}{(x-2)^2(x^2+1)} = \frac{-2}{x-2} + \frac{4}{(x-2)^2} + \frac{2x}{x^2+1}$$

As an alternate to (4), (6) and (7), differentiate (1):

$$6 = A(x^2+1) + A(x-2)2x + B2x + C(x-2)^2 + (Cx+D)2(x-2) \qquad (8)$$

and plug in $x = 2$: $\quad 6 = A(5) + B(4) = 5A + 16 \quad \Rightarrow \quad 5A = -10$

$$A = -2 \qquad (9)$$

12. (20 points) Find the 4th degree Maclaurin polynomial (Taylor polynomial at 0) for $f(x) = \sec x$. Show your derivation on paper. (Yes, you need 4 derivatives.) Enter your coefficients here.

$$\sec x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

Solution:

$$\begin{aligned}
f(x) &= \sec x & f(0) &= 1 \\
f'(x) &= \sec x \tan x & f'(0) &= 0 \\
f''(x) &= \sec x \tan^2 x + \sec^3 x & f''(0) &= 1 \\
f^{(3)}(x) &= \sec x \tan^3 x + 5\sec^3 x \tan x & f^{(3)}(0) &= 0 \\
f^{(4)}(x) &= \sec x \tan^4 x + 18\sec^3 x \tan^2 x + 5\sec^5 x & f^{(4)}(0) &= 5
\end{aligned}$$

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \frac{1}{4!}f^{(4)}(0)x^4$$

$$\sec x = 1 + \frac{1}{2!}x^2 + \frac{5}{4!}x^4$$

$c_0 =$ __1__ $\quad c_1 =$ __0__ $\quad c_2 =$ __$\frac{1}{2}$__ $\quad c_3 =$ __0__ $\quad c_4 =$ __$\frac{5}{24}$__