MATH 172 EXAM 1 Fall 1998

Section 502 Solutions P. Yasskin

Multiple Choice: (4 points each)

1. Evaluate $\int_0^2 \sqrt{4-x^2}+1\,dx$ by interpretating it as an area.

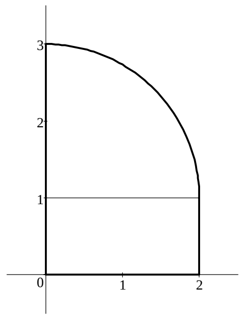
a. $2+2\pi$

b. $1+\pi$

c. $1+2\pi$

d. $\pi-1$

e. $2+\pi\ldots$ correctchoice



This is a quarter circle on top of a rectangle. So

$$\int_0^2 \sqrt{4-x^2}+1\,dx = \frac{1}{4}\pi r^2 + LW$$
$$= \frac{1}{4}\pi(2)^2 + 2\bullet 1 = \pi+2$$

2. Approximate the area between the curves $y=x$ and $y=1+x^2$ for $0\le x\le 6$ using 3 rectangles with equal widths and with heights given by the function values at the **left** endpoints.

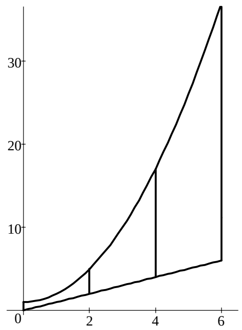
a. 17

b. 29

c. $34\ldots$ correctchoice

d. 46

e. 58



The widths are each 2.

The heights a gotten by subtracting the function values at 0, 2 and 4:

$$A = 2(1-0)+2(5-2)+2(17-4) = 34$$

3. A girl walks(runs) in a straight line with acceleration $a(t) = 4t + \sin t$. If her initial velocity is $v(0) = 3$, find her velocity at $t = 2$.

- **a.** $12 - \cos 2 \ldots$ correctchoice
- **b.** $12 + \cos 2$
- **c.** $10 - \cos 2$
- **d.** $10 + \cos 2$
- **e.** $8 + \sin 2$

$a(t) = 4t + \sin t \qquad v(t) = \int 4t + \sin t\, dt = 2t^2 - \cos t + C \qquad v(0) = -1 + C = 3$

$C = 4 \qquad v(t) = 2t^2 - \cos t + 4 \qquad v(2) = 12 - \cos 2$

4. Compute: $\int_0^1 x^{3/7}\, dx$

- **a.** $-\frac{7}{4}$
- **b.** $-\frac{4}{7}$
- **c.** $\frac{7}{3}$
- **d.** $\frac{3}{7}$
- **e.** $\frac{7}{10} \ldots$ correctchoice

$\int_0^1 x^{3/7}\, dx = \left[\frac{7x^{10/7}}{10} \right]_0^1 = \frac{7}{10}$

5. Compute: $\int \sqrt{x} \left(x^2 - \frac{1}{x} \right) dx$

- **a.** $\frac{2x^{7/2}}{7} + \frac{2x^{-3/2}}{3} + C$
- **b.** $\frac{2x^{7/2}}{7} - 2x^{1/2} + C \ldots$ correctchoice
- **c.** $\frac{2x^{37/2}}{3} + \frac{2x^{-3/2}}{3} + C$
- **d.** $\sqrt{x} \left(\frac{x^3}{3} - \ln x \right) + \frac{2x^{3/2}}{3}(x^2 - \ln x) + C$
- **e.** $\frac{2x^{3/2}}{3} \left(\frac{x^3}{3} - \ln x \right) + C$

$\int \sqrt{x} \left(x^2 - \frac{1}{x} \right) dx = \int (x^{5/2} - x^{-1/2})\, dx = \frac{2x^{7/2}}{7} - 2x^{1/2} + C$

6. Compute: $\int_0^2 x\sqrt{4-x^2}\,dx$

a. $\frac{2\sqrt{2}}{3}$

b. $\frac{8}{3}$... correctchoice

c. 24

d. $\frac{32}{3}$

e. 6

$u = 4 - x^2 \qquad du = -2x\,dx \qquad -\frac{1}{2}du = x\,dx \qquad$ $x = 2$ at $u = 0$, $x = 0$ at $u = 4$

$$\int_0^2 x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int_4^0 u^{1/2}du = -\frac{1}{2}\left[\frac{2u^{3/2}}{3}\right]_4^0 = -\frac{1}{2}[0] + \frac{1}{2}\left[\frac{2(4)^{3/2}}{3}\right] = \frac{8}{3}$$

7. Compute: $\int_0^{1/4} \sin(\pi t)\,dt$

a. $\frac{1}{\pi\left(\sqrt{2}-1\right)}$

b. $\frac{1}{\pi\left(1-\sqrt{2}\right)}$

c. $\frac{1}{\pi\sqrt{2}} - \frac{1}{\pi}$

d. $\frac{1}{\pi} - \frac{1}{\pi\sqrt{2}}$... correctchoice

e. $-\frac{1}{\pi\sqrt{2}}$

$u = \pi t \qquad du = \pi\,dt \qquad \frac{1}{\pi}du = dt$

$$\int_0^{1/4} \sin(\pi t)\,dt = \frac{1}{\pi}\int \sin u\;du = \frac{1}{\pi}[-\cos u] = -\frac{1}{\pi}\left[\cos(\pi t)\right]_0^{1/4}$$

$$= -\frac{1}{\pi}\left[\cos\frac{\pi}{4}\right] + \frac{1}{\pi}[\cos 0] = \frac{1}{\pi} - \frac{1}{\pi\sqrt{2}}$$

8. The mass density of a 3 cm bar is $\rho = 1 + x^2\ \frac{\text{gm}}{\text{cm}}$ for $0 \le x \le 3$. Find the total mass of the bar.

- **a**. 4 gm
- **b**. 10 gm
- **c**. $12\text{ gm} \ldots$ correctchoice
- **d**. 18 gm
- **e**. 30 gm

$$M = \int_0^3 1 + x^2\,dx = \left[x + \frac{x^3}{3} \right]_0^3 = 3 + 9 = 12$$

9. The mass density of a 3 cm bar is $\rho = 1 + x^2\ \frac{\text{gm}}{\text{cm}}$ for $0 \le x \le 3$. Find the average density of the bar.

- **a**. $4\ \frac{\text{gm}}{\text{cm}} \ldots$ correctchoice
- **b**. $10\ \frac{\text{gm}}{\text{cm}}$
- **c**. $12\ \frac{\text{gm}}{\text{cm}}$
- **d**. $\frac{10}{3}\ \frac{\text{gm}}{\text{cm}}$
- **e**. $\frac{13}{4}\ \frac{\text{gm}}{\text{cm}}$

$$\rho_{\text{ave}} = \frac{1}{3}\int_0^3 1 + x^2\,dx = \frac{M}{3} = \frac{12}{3} = 4$$

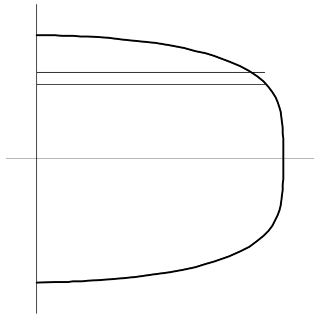
10. The mass density of a 3 cm bar is $\rho = 1 + x^2\ \frac{\text{gm}}{\text{cm}}$ for $0 \le x \le 3$. Find the x-coordinate of the center of mass of the bar.
(If you prefer, you may think of this as a plate of uniform density $\rho = 1$ between $y = 1 + x^2$ and the x-axis for $0 \le x \le 3$.)

- **a**. $\frac{3}{2}$
- **b**. 2
- **c**. $\frac{7}{3}$
- **d**. $\frac{33}{16} \ldots$ correctchoice
- **e**. $\frac{99}{4}$

$$1^{\text{st}}\text{mom} = \int_0^3 x(1 + x^2)\,dx = \int_0^3 x + x^3\,dx = \left[\frac{x^2}{2} + \frac{x^4}{4} \right]_0^3$$

$$= \frac{9}{2} + \frac{81}{4} = \frac{18 + 81}{4} = \frac{99}{4} \qquad \bar{x} = \frac{1^{\text{st}}\text{mom}}{M} = \frac{99}{4}\,\frac{1}{12} = \frac{33}{16}$$

11.



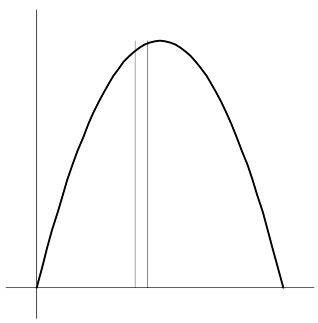
(15 points) The area between the curve $x = \sqrt{16 - y^4}$ and the y-axis is rotated about the y-axis. Find the volume of the solid swept out.

This is a y-integral. The limits occur when $x = 0$. So $y^4 = 16$ or $y = \pm 2$. Cutting up the y-axis, drawing a rectangle and rotating, gives a disk of radius $x = \sqrt{16 - y^4}$. So the volume is

$$V = \int_{-2}^{2} \pi \left(\sqrt{16 - y^4} \right)^2 dy = \int_{-2}^{2} \pi (16 - y^4)\, dy = \pi \left[16y - \frac{y^5}{5} \right]_{-2}^{2}$$

$$= 2\pi \left[16 \bullet 2 - \frac{(2)^5}{5} \right] = 64\pi \left(1 - \frac{1}{5} \right) = \frac{256\pi}{5}$$

12.



(15 points) The area between the curve $y = 4x - x^2$ and the x-axis is rotated about the y-axis. Find the volume of the solid swept out.

This is an x-integral. The limits occur when $y = 0$. So $x(4 - x) = 0$ or $x = 0, 4$. Cutting up the x-axis, drawing a rectangle and rotating, gives a cylinder of radius x and height $y = 4x - x^2$. So the volume is

$$V = \int_{0}^{4} 2\pi x (4x - x^2)\, dx = \int_{0}^{4} 2\pi (4x^2 - x^3)\, dx = 2\pi \left[\frac{4x^3}{3} - \frac{x^4}{4} \right]_{0}^{4}$$

$$= 2\pi \left[\frac{256}{3} - \frac{256}{4} \right] = 512\pi \left(\frac{1}{3} - \frac{1}{4} \right) = 512\pi \left(\frac{1}{12} \right) = \frac{128\pi}{3}$$

13. (15 points) Find the arc length of the parametric curve $x = \frac{1}{2}t^6, \quad y = t^4$ between $t = 0$ and $t = 1$.

HINT: $\sqrt{t^{2a} + t^{2a+b}} = t^a\sqrt{1 + t^b}$

$$\frac{dx}{dt} = 3t^5 \qquad \frac{dy}{dt} = 4t^3$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3t^5)^2 + (4t^3)^2} = \sqrt{9t^{10} + 16t^6} = t^3\sqrt{9t^4 + 16}$$

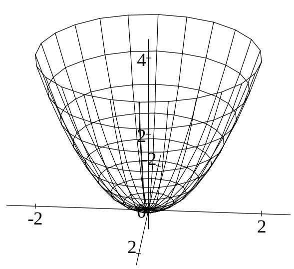
$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 t^3\sqrt{9t^4 + 16}\, dt$$

$$u = 9t^4 + 16 \qquad du = 36t^3\, dt \qquad \frac{1}{36}du = t^3\, dt$$

$$L = \frac{1}{36}\int_{t=0}^1 u^{1/2}\, du = \frac{1}{36}\left[\frac{2u^{3/2}}{3}\right]_{t=0}^1 = \frac{1}{36}\left[\frac{2(9t^4 + 16)^{3/2}}{3}\right]_{t=0}^1$$

$$= \frac{1}{36}\left[\frac{2}{3}(25)^{3/2} - \frac{2}{3}(16)^{3/2}\right] = \frac{1}{54}[125 - 64] = \frac{61}{54}$$

14.



(15 points) A bowl is formed by rotating the curve $y = x^2$ for $0 \le x \le 2$ about the y-axis. This bowl is full of water. How much work is done in pumping the water out the top of the bowl? Leave the density as ρ and the acceleration of gravity as g.

This is a y-integral for $0 \le y \le 4$. A slice perpenducular to the y-axis at height y is a thin disk of radius $x = \sqrt{y}$. So its volume is $dV = \pi\left(\sqrt{y}\right)^2 dy$, it mass is $dm = \rho\, dV = \rho\pi y\, dy$ and its weight is $dF = dm\, g = \rho g\pi y\, dy$. This weight must be lifted a distance $D = 4 - y$. So the work done is

$$W = \int_0^4 D\, dF = \int_0^4 \rho g\pi y(4 - y)\, dy = \rho g\pi\int_0^4 4y - y^2\, dy = \rho g\pi\left[2y^2 - \frac{y^3}{3}\right]_0^4$$

$$= \rho g\pi\left[32 - \frac{64}{3}\right]_0^4 = 32\rho g\pi\left[1 - \frac{2}{3}\right]_0^4 = \frac{32\rho g\pi}{3}$$