Name________________________________ ID____________________

MATH 172 FINAL EXAM Fall 1998

Section 502 Solutions P. Yasskin

Multiple Choice: (8 points each)

**1.** Compute $\int_0^{\sqrt{\pi}} x\sin(x^2)\ dx$

**a.** $-\frac{1}{2}$

**b.** $0$

**c.** $\frac{1}{2}$

**d.** $1$ correctchoice

**e.** $2$

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\ dx = \left.\frac{-\cos(x^2)}{2}\right|_0^{\sqrt{\pi}} = \frac{-\cos(\pi)}{2} - \frac{-\cos(0)}{2} = \frac{--1}{2} - \frac{-1}{2} = 1$$

**2.** Compute $\int_0^1 x^2\ e^x\ dx$

**a.** $-3e$

**b.** $-3e+2$

**c.** $-3e-2$

**d.** $e$

**e.** $e-2$ correctchoice

$$u = x^2 \qquad dv = e^x\,dx$$
$$du = 2x\,dx \qquad v = e^x$$

$$\int_0^1 x^2\ e^x\ dx = \left[x^2e^x - 2\int x\ e^x\ dx\right]_0^1$$

$$u = x \qquad dv = e^x\,dx$$
$$du = dx \qquad v = e^x$$

$$= \left[x^2e^x - 2\left(xe^x - \int e^x\ dx\right)\right]_0^1$$

$$= \left[x^2e^x - 2(xe^x - e^x)\ \right]_0^1 = [e-2(e-e)] - [0-2(0-1)] = e-2$$

**3.** Find the average value of the function $f(x) = 3x^2+1$ for $1 \le x \le 3.$

**a.** $13$

**b.** $14$ correctchoice

**c.** $15$

**d.** $16$

**e.** $17$

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{2}\int_1^3 (3x^2+1)\,dx = \left[\frac{1}{2}(x^3+x)\right]_1^3 = \left[\frac{27+3}{2}\right] - \left[\frac{1+1}{2}\right] = 14$$

**4**. Compute $\int \frac{x^2}{(1-x^2)^{3/2}}\, dx$

Hints: $\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$

**a**. $\frac{x}{\sqrt{1-x^2}} - \arctan x$

**b**. $\frac{x}{\sqrt{1-x^2}} + \arctan x$

**c**. $\frac{x}{\sqrt{1-x^2}} - \arcsin x$ correctchoice

**d**. $\frac{x}{\sqrt{1-x^2}} + \arcsin x$

**e**. $\frac{x}{\sqrt{1-x^2}} + x$

Trig sub: $x = \sin\theta \qquad dx = \cos\theta\, d\theta \qquad \cos\theta = \sqrt{1-x^2}$

$$\int \frac{x^2}{(1-x^2)^{3/2}}\, dx = \int \frac{\sin^2\theta}{\cos^3\theta}\cos\theta\, d\theta = \int \tan^2\theta\, d\theta = \int \sec^2\theta - 1\, d\theta = [\tan\theta - \theta]$$

$$= \frac{x}{\sqrt{1-x^2}} - \arcsin x$$

**5**. Find the area between the curves $y = x^2$ and $y = x^3$ for $0 \le x \le 1$.

**a**. $\frac{1}{24}$

**b**. $\frac{1}{12}$ correctchoice

**c**. $\frac{1}{7}$

**d**. $\frac{1}{6}$

**e**. $1$

$$A = \int_0^1 (x^2 - x^3)\, dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

**6**. The area between the curves $y = x^2$ and $y = x^3$ for $0 \le x \le 1$ is rotated around the $x$-axis. Find the volume swept out.

**a**. $\frac{\pi}{5}$

**b**. $\frac{\pi}{10}$

**c**. $\frac{\pi}{20}$

**d**. $\frac{2\pi}{35}$ correctchoice

**e**. $\frac{4\pi}{35}$

Washers:

$$V = \int \pi R^2 - \pi r^2\, dx = \int_0^1 \pi(x^2)^2 - \pi(x^3)^2\, dx = \left[ \frac{\pi x^5}{5} - \frac{\pi x^7}{7} \right]_0^1 = \frac{\pi}{5} - \frac{\pi}{7} = \frac{2\pi}{35}$$

**7.** Find the total mass of a $6$ cm bar whose linear density is $\rho = (1 + x)$ g/cm where $x$ is the distance from one end in cm.

**a.** $\frac{7}{6}$ g

**b.** $4$ g

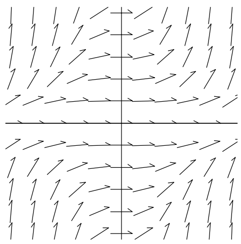
**c.** $6$ g

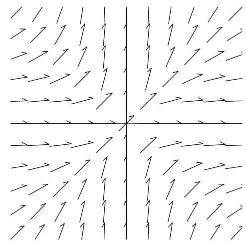
**d.** $7$ g

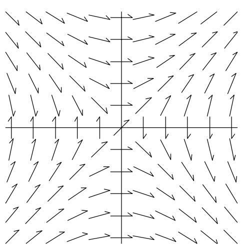
**e.** $24$ g correctchoice

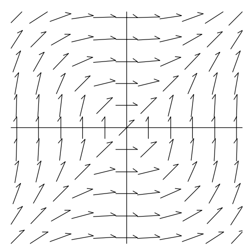
$$M = \int \rho\, dx = \int_0^6 (1 + x)\, dx = \left[ x + \frac{x^2}{2} \right]_0^6 = 6 + \frac{36}{2} = 24$$

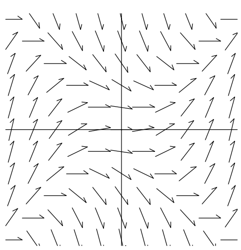
**8.** Which of the following is the direction field for the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$ ?

**a.**  No, vert at $y = 0$

**d**  No, hor at $x = 0$

**b.**  No,all pos slope

**e.**  correctchoice

**c.**  No,all pos slope

**9**. Compute $\int_1^{\infty} \frac{1}{1+x^2}\, dx$

**a.** $\frac{\pi}{4}$ correctchoice

**b.** $\frac{\pi}{2}$

**c.** $\frac{3\pi}{4}$

**d.** Convergent but none of the above

**e.** Divergent

$$\int_1^{\infty} \frac{1}{1+x^2}\, dx = \arctan x \Big|_1^{\infty} = \arctan\infty - \arctan 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

**10**. Compute $\sum_{n=1}^{\infty} \frac{1}{1+n^2} =$

**a.** $\frac{\pi}{4}$

**b.** $\frac{\pi}{2}$

**c.** $\frac{3\pi}{4}$

**d.** Convergent but none of the above correctchoice

**e.** Divergent

Since $\int_1^{\infty} \frac{1}{1+x^2}\, dx = \frac{\pi}{4}$ is convergent, so is $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$.

Since $\int_0^{\infty} \frac{1}{1+x^2}\, dx = \frac{\pi}{2}$ also, $\frac{\pi}{4} < \sum_{n=1}^{\infty} \frac{1}{1+n^2} < \frac{\pi}{2}$

**11**. The series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{1+n^2}$ is

**a.** absolutely convergent

**b.** conditionally convergent correctchoice

**c.** divergent

**d.** none of these

$\sum_{n=1}^{\infty} (-1)^n \frac{n}{1+n^2}$ is convergent because it is an alternating, decreasing series and $\lim_{n\to\infty} \frac{n}{1+n^2} = 0$ However, the related absolute series $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$ is divergent because $\int_1^{\infty} \frac{x}{1+x^2}\, dx = \frac{1}{2}\ln(1+x^2)\Big|_1^{\infty} = \infty$.

**12**. Find the radius of convergence of the series $\sum_{n=2}^{\infty} \frac{(x-3)^n}{2^n n^2}$.

**a**. 0
**b**. 1
**c**. 2 correctchoice
**d**. 3
**e**. 4

$$a_n = \frac{(x-3)^n}{2^n n^2} \qquad a_{n+1} = \frac{(x-3)^{n+1}}{2^{n+1}(n+1)^2}$$

$$\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{2^{n+1}(n+1)^2}\frac{2^n n^2}{(x-3)^n}\right| = \frac{|x-3|}{2}\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right| = \frac{|x-3|}{2} < 1 \qquad |x-3| < 2 = R$$

**13**. The vectors $\vec{a} = (2,-1,4)$ and $\vec{b} = (3,2,-1)$ are

**a**. parallel
**b**. perpendicular correctchoice
**c**. neither

$\vec{a} \bullet \vec{b} = 6 - 2 - 4 = 0$

**14**. Find the area of the parallelogram whose edges are $\vec{a} = (1,2,3)$ and $\vec{b} = (3,2,1)$.

**a**. $2\sqrt{2}$
**b**. $4\sqrt{2}$
**c**. $4\sqrt{3}$
**d**. $2\sqrt{6}$
**e**. $4\sqrt{6}$ correctchoice

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = \hat{\imath}(2-6) - \hat{\jmath}(1-9) + \hat{k}(2-6) = (-4,8,-4)$$

$$A = \left|\vec{a}\times\vec{b}\right| = \sqrt{16+64+16} = \sqrt{96} = 4\sqrt{6}$$

**15**. A baseball is thrown straight North initially at $45^\circ$ above horizontal and follows a parabolic path, up and back down. At the top of the trajectory, in what direction does the unit binormal $\hat{B}$ point?

**a**. West and horizontal correctchoice
**b**. West and below horizontal
**c**. Straight down
**d**. East and below horizontal
**e**. East and horizontal

$\hat{T}$ is North and horizontal. $\bar{N}$ is straight down.
So by the right hand rule, $\hat{B}$ is West and horizontal.

## Work-out Problems: (20 points each)

**16.** Solve the differential equation $\dfrac{dy}{dx} + 2xy = e^{-x^2}$ with the initial condition $y(1) = 0$.

Linear: $P = 2x \qquad Q = e^{-x^2} \qquad I = e^{\int P dx} = e^{\int 2x dx} = e^{x^2}$

$e^{x^2}\dfrac{dy}{dx} + e^{x^2}2xy = e^{x^2}e^{-x^2} \qquad \left(e^{x^2}y\right)' = 1 \qquad e^{x^2}y = \int 1 dx = x + C$

$x = 1$ when $y = 0$ : $\quad e^1 0 = 1 + C \quad$ So $\quad C = -1 \qquad e^{x^2}y = x - 1 \qquad y = (x-1)e^{-x^2}$

**17.** Find the point where the line $\left\{\begin{array}{l} x = -2 + t \\ y = 1 + 2t \\ z = 3 - 2t \end{array}\right\}$ intersects the plane $2x - 3y + z = -16$.

Plug the line into the plane and solve for $t$:

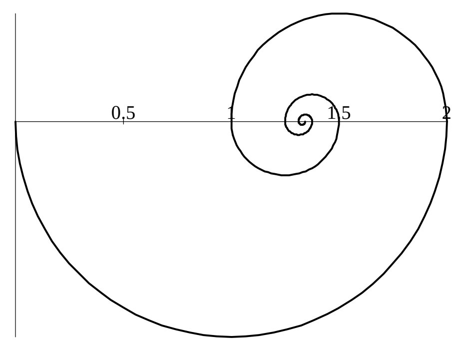
$2(-2+t) - 3(1+2t) + (3-2t) = -16 \qquad -4 - 6t = -16 \qquad -6t = -12 \qquad t = 2$

Plug back into the line:

$x = -2 + t = -2 + 2 = 0 \qquad y = 1 + 2t = 1 + 4 = 5 \qquad z = 3 - 2t = 3 - 4 = -1$

So the point is $\quad (x,y,z) = (0,5,-1)$

**18.** The spiral at the right is made from an infinite number of semicircles whose centers are all on the $x$-axis. The radius of each semicircle is half of the radius of the previous semicircle.




**a.** Consider the infinite sequence of points where the spiral crosses the $x$-axis. What is the $x$-coordinate of the limit of this sequence?

$$2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots = \sum_{n=0}^{\infty} 2\left(-\frac{1}{2}\right)^n = \frac{2}{1 + \frac{1}{2}} = \frac{4}{3}$$

**b.** What is the total length of the spiral (with an infinite number of semicircles)? Or, is the length infinite?

Each semicircle has length $\quad L_n = \pi r_n \quad$ where the radii are $r_n = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$.

So the total length is $\quad L = \displaystyle\sum_{n=0}^{\infty} \pi r_n = \sum_{n=0}^{\infty} \pi\left(\frac{1}{2}\right)^n = \frac{\pi}{1 - \frac{1}{2}} = 2\pi$.

**19**. Consider the twisted cubic curve $\vec{r}(t) = (6t, 3t^2, t^3)$ for $0 \le t \le 2$.

**a**. Find the arc length of the curve between $t = 0$ and $t = 2$.

$$\vec{v} = (6, 6t, 3t^2) \qquad |\vec{v}| = \sqrt{36 + 36t^2 + 9t^4} = \sqrt{(6 + 3t^2)^2} = 6 + 3t^2$$

$$L = \int ds = \int |\vec{v}|\,dt = \int_0^2 (6 + 3t^2)\,dt = \Big[\, 6t + t^3 \,\Big]_0^2 = 12 + 8 = 20$$

**b**. Find the unit binormal vector $\hat{B}$.

$$\vec{a} = (0, 6, 6t)$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 6t & 3t^2 \\ 0 & 6 & 6t \end{vmatrix} = \hat{\imath}(36t^2 - 18t^2) - \hat{\jmath}(36t - 0) + \hat{k}(36 - 0)$$

$$= (18t^2, -36t, 36) = 18(t^2, -2t, 2)$$

$$|\vec{v} \times \vec{a}| = 18\sqrt{t^4 + 4t^2 + 4} = 18\sqrt{(t^2 + 2)^2} = 18(t^2 + 2)$$

$$\hat{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \frac{18}{18(t^2 + 2)}(t^2, -2t, 2) = \left( \frac{t^2}{t^2 + 2}, \frac{-2t}{t^2 + 2}, \frac{2}{t^2 + 2} \right)$$