MATH 172 EXAM 2 Spring 1999

Section 504 Solutions P. Yasskin

1. Approximate $\int_1^7 (x^2+2x)\,dx$ by using the trapezoid rule with 3 intervals.

- **a.** 83
- **b.** 116
- **c.** 162
- **d.** 166 correctchoice
- **e.** 232

$f(x) = x^2 + 2x \qquad \Delta x = \frac{7-1}{3} = 2$

$$\int_1^7 (x^2+2x)\,dx \approx \left[\tfrac{1}{2}f(1) + f(3) + f(5) + \tfrac{1}{2}f(7)\right]\Delta x = f(1) + 2f(3) + 2f(5) + f(7)$$

$$= (3) + 2(15) + 2(35) + (49+14) = 166$$

2. Given that the partial fraction expansion for $\frac{x^2-1}{x^4+x^2}$ is $\frac{x^2-1}{x^4+x^2} = \frac{2}{x^2+1} - \frac{1}{x^2}$, compute $\int \frac{x^2-1}{x^4+x^2}\,dx$.

- **a.** $2\tan^{-1}x + \frac{1}{x} + C$ correctchoice
- **b.** $\tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{x} + C$
- **c.** $\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{x} + C$
- **d.** $\frac{-4x}{(x^2+1)^2} + \frac{2}{x^3} + C$
- **e.** None of These

$$\int \frac{x^2-1}{x^4+x^2}\,dx = \int \frac{2}{x^2+1} - \frac{1}{x^2}\,dx = 2\tan^{-1}x + \frac{1}{x} + C$$

3. Compute: $\displaystyle\int_1^2 \frac{1}{\sqrt{x^2-1}}\,dx$

a. $\sin^{-1}(2) - \frac{\pi}{2}$

b. $\frac{\pi}{2} - \sin^{-1}(2)$

c. $\ln\left(2+\sqrt{3}\right)$ correctchoice

d. $\ln\left(2-\sqrt{3}\right)$

e. $\ln\left(\frac{\sec 2 + \tan 2}{\sec 1 + \tan 1}\right)$

$x \ge 1$ So: $x = \sec\theta$ $dx = \sec\theta\tan\theta\,d\theta$

$$\int_1^2 \frac{1}{\sqrt{x^2-1}}\,dx = \int_{x=1}^2 \frac{1}{\sqrt{\sec^2\theta - 1}}\sec\theta\tan\theta\,d\theta = \int_{x=1}^2 \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$$

$$= \left[\ln\left|x + \sqrt{x^2-1}\right|\right]_1^2 = \ln\left(2+\sqrt{3}\right) - \ln\left(1+\sqrt{0}\right) = \ln\left(2+\sqrt{3}\right)$$

4. The improper integral $\displaystyle\int_1^2 \frac{1}{(x-1)^2}\,dx$

a. diverges to $-\infty$

b. converges to a negative number

c. converges to 0

d. converges to a positive number

e. diverges to $+\infty$ correctchoice

$$\int_1^2 \frac{1}{(x-1)^2}\,dx = \left[-\frac{1}{x-1}\right]_1^2 = \left[-\frac{1}{2-1}\right] - \lim_{x\to 1^+}\left[-\frac{1}{x-1}\right] = -1+\infty = \infty$$

5. The improper integral $\displaystyle\int_2^\infty \frac{1}{x^2+1+\sin x}\,dx$

a. diverges to $-\infty$

b. converges and is $< \frac{1}{2}$ correctchoice

c. converges and is $= \frac{1}{2}$

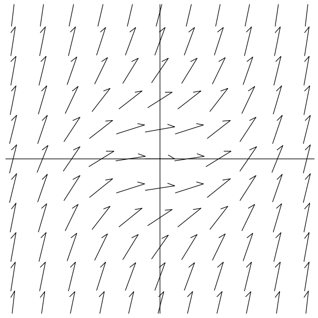
d. converges and is $> \frac{1}{2}$

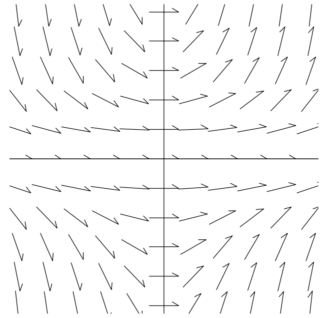
e. diverges to $+\infty$

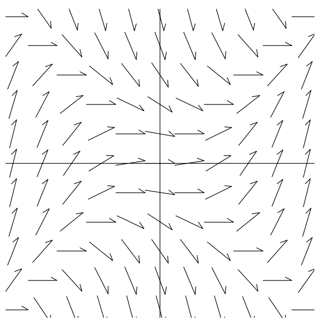
$-1 \le \sin x \le 1$ $x^2 \le x^2+1+\sin x \le x^2+2$ $\frac{1}{x^2} \ge \frac{1}{x^2+1+\sin x} \ge \frac{1}{x^2+2}$

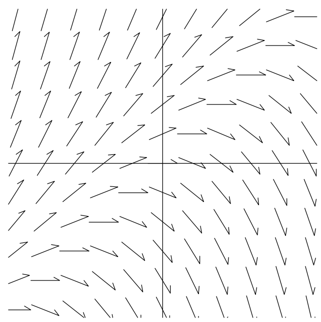
So $\displaystyle\int_2^\infty \frac{1}{x^2+1+\sin x}\,dx \le \int_2^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_2^\infty = \frac{1}{2}$

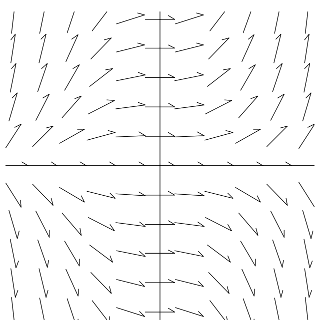
6. Which of the following is the direction field of the differential equation $\frac{dy}{dx} = xy^2$?

a. 

d.  correctchoice

b. 

e. 

c. 

$\frac{dy}{dx} = 0$ on both axes

$\frac{dy}{dx} > 0$ when $x > 0$

$\frac{dy}{dx} < 0$ when $x < 0$

7. Solve the initial value problem $\frac{dy}{dx} = xy^2$ with the initial condition $y(1) = \frac{2}{5}$. Then find $y(2)$.

a. 0

b. $\frac{1}{5}$

c. $\frac{4}{5}$

d. $\frac{7}{10}$

e. 1 correctchoice

Separate variables: $\int \frac{1}{y^2}\,dy = \int x\,dx \qquad -\frac{1}{y} = \frac{x^2}{2} + C$

Initial Condition ($x = 1, y = \frac{2}{5}$): $\quad -\frac{5}{2} = \frac{1}{2} + C \qquad C = -3$

Solve: $\quad -\frac{1}{y} = \frac{x^2}{2} - 3 \qquad \frac{1}{y} = 3 - \frac{x^2}{2} = \frac{6 - x^2}{2} \qquad y = \frac{2}{6 - x^2}$

Evaluate: $\quad y(2) = \frac{2}{6 - 2^2} = 1$

8. The mass density of a 3 ft bar is $\rho = 1 + x^2 \ \frac{\text{lb}}{\text{ft}}$ for $0 \le x \le 3$. Find the center of mass of the bar.

- **a**. $\bar{x} = 12$
- **b**. $\bar{x} = \frac{16}{33}$
- **c**. $\bar{x} = \frac{33}{16}$ correctchoice
- **d**. $\bar{x} = \frac{4}{99}$
- **e**. $\bar{x} = \frac{99}{4}$

$$M = \int_0^3 \rho\,dx = \int_0^3 (1 + x^2)\,dx = \left[x + \frac{x^3}{3} \right]_0^3 = 3 + \frac{27}{3} = 12$$

$$mom = \int_0^3 x\rho\,dx = \int_0^3 x(1 + x^2)\,dx = \int_0^3 x + x^3\,dx = \left[\frac{x^2}{2} + \frac{x^4}{4} \right]_0^3 = \frac{9}{2} + \frac{81}{4} = \frac{99}{4}$$

$$\bar{x} = \frac{mom}{M} = \frac{99}{4}\frac{1}{12} = \frac{33}{16}$$

9. Find the arc length of the parametric curve $x = 2t^2$ and $y = t^3 + 3$ between $t = 0$ and $t = 1$.

- **a**. $\frac{61}{27}$ correctchoice
- **b**. $\frac{125}{9}$
- **c**. $\frac{125}{27}$
- **d**. $\frac{122}{3}$
- **e**. $\frac{250}{3}$

$$\frac{dx}{dt} = 4t \qquad \frac{dy}{dt} = 3t^2$$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_0^1 \sqrt{16t^2 + 9t^4}\ dt = \int_0^1 t\sqrt{16 + 9t^2}\ dt$$

$$= \left[\frac{2}{3}\frac{1}{18}(16 + 9t^2)^{3/2} \right]_0^1 = \frac{1}{27}(25)^{3/2} - \frac{1}{27}(16)^{3/2} = \frac{1}{27}(125 - 64) = \frac{61}{27}$$

10. The curve $y = x^3$ for $0 \le x \le 2$ is rotated about the x-axis. The area of the resulting surface may be computed from the integral

a. $\int_0^2 \pi x\sqrt{1+9x^4}\,dx$

b. $\int_0^2 2\pi x^3\sqrt{1+9x^4}\,dx$ correctchoice

c. $\int_0^2 2\pi x\sqrt{1+x^6}\,dx$

d. $\int_0^2 \pi x^3\sqrt{1+x^6}\,dx$

e. $\int_0^2 2\pi x\sqrt{1+9x^4}\,dx$

$$A = \int_0^2 (\text{circumference})\,ds = \int_0^2 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 2\pi x^3\sqrt{1+9x^4}\,dx$$

11. (10 points) Find the partial fraction expansion for $\dfrac{2x^2 - x + 2}{x^3 + x}$.
(Do not integrate. HINT: Try $x = 0, 1, -1$.)

$$\frac{2x^2 - x + 2}{x^3 + x} = \frac{2x^2 - x + 2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + C}{x^2+1} \qquad 2x^2 - x + 2 = A(x^2+1) + (Bx + C)x$$

$x = 0:$ $\quad 2 = A$

$x = 1:$ $\quad 3 = A(2) + (B + C) = 4 + B + C \quad \Rightarrow \quad B + C = -1$

$x = -1:$ $\quad 5 = A(2) + (-B + C)(-1) = 4 + B - C \quad \Rightarrow \quad B - C = 1$

$A = 2 \qquad B = 0 \qquad C = -1$

$$\frac{2x^2 - x + 2}{x^3 + x} = \frac{2}{x} + \frac{-1}{x^2+1}$$

12. (10 points) Compute: $\int \dfrac{1}{(1-x^2)^{3/2}}\,dx$.

$x = \sin\theta \qquad dx = \cos\theta\,d\theta$

$$\int \frac{1}{(1-x^2)^{3/2}}\,dx = \int \frac{1}{\left(1-\sin^2\theta\right)^{3/2}}\cos\theta\,d\theta = \int \frac{1}{\cos^3\theta}\cos\theta\,d\theta$$

$$= \int \frac{1}{\cos^2\theta}\,d\theta = \int \sec^2\theta\,d\theta = \tan\theta + C = \frac{\sin\theta}{\cos\theta} + C = \frac{x}{\sqrt{1-x^2}} + C$$

13. (10 points) Solve the initial value problem $\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{3x^2}{1+x^2}$ with the initial condition $y(1) = 2$.

$$P = \frac{2x}{1+x^2} \qquad \int P\,dx = \int \frac{2x}{1+x^2}\,dx = \ln(1+x^2) \qquad I = e^{\int P\,dx} = e^{\ln(1+x^2)} = 1+x^2$$

$$(1+x^2)\frac{dy}{dx} + 2xy = 3x^2 \qquad \frac{d}{dx}\left[(1+x^2)y\right] = 3x^2 \qquad (1+x^2)y = \int 3x^2\,dx = x^3 + C$$

Initial condition ($x = 1, y = 2$): $(1+1)2 = 1 + C \qquad C = 3$

$$(1+x^2)y = x^3 + 3 \qquad y = \frac{x^3+3}{1+x^2}$$

14. (10 points) A nuclear power plant went on line at the beginning of the year 1980. It has produced isotope X at the rate of $10\ \frac{\text{kg}}{\text{yr}}$ and the half-life of X is 20 yr. (So its decay constant is $k = \frac{\ln 2}{20}$.) The plant stores all of the isotope X it produces. If there was no isotope X at the beginning of 1980, how much isotope X will there be at the beginning of the year 2000? (6 points for the equations.)

Let $X(t)$ be the kg of X at time t. Solve: $\frac{dX}{dt} = 10 - \frac{\ln 2}{20}X$ with $X(0) = 0$.

Separate variables: $\int \frac{dX}{10 - \frac{\ln 2}{20}X} = \int dt \qquad -\frac{20}{\ln 2}\ln\left|10 - \frac{\ln 2}{20}X\right| = t + C$

Use initial conditions ($t = 0,\ X = 0$): $C = -\frac{20}{\ln 2}\ln 10$

Solve for X: $-\frac{20}{\ln 2}\ln\left|10 - \frac{\ln 2}{20}X\right| = t - \frac{20}{\ln 2}\ln 10$

$$\ln\left|10 - \frac{\ln 2}{20}X\right| = -\frac{\ln 2}{20}t + \ln 10 \qquad \left|10 - \frac{\ln 2}{20}X\right| = e^{\ln 10}e^{-\frac{t\ln 2}{20}} = 10e^{-\frac{t\ln 2}{20}}$$

$$10 - \frac{\ln 2}{20}X = e^{\ln 10}e^{-\frac{t\ln 2}{20}} = \pm 10e^{-\frac{t\ln 2}{20}} \qquad \frac{\ln 2}{20}X = 10 \pm 10e^{-\frac{t\ln 2}{20}} \qquad X = \frac{200}{\ln 2}\left(1 - e^{-\frac{t\ln 2}{20}}\right)$$

The minus is needed to give the initial condition. The year 2000 is $t = 20$. So

$$X(20) = \frac{200}{\ln 2}\left(1 - e^{-\frac{20\ln 2}{20}}\right) = \frac{200}{\ln 2}(1 - e^{-\ln 2}) = \frac{200}{\ln 2}\left(1 - \frac{1}{2}\right) = \frac{100}{\ln 2} \approx 144\text{kg}$$