MATH 172 EXAM 3 Spring 1999

Section 504 Solutions P. Yasskin

Multiple Choice: (6 points each)

1. Compute $\lim_{n\to\infty} \frac{\ln n}{n}$

a. $-\infty$

b. -1

c. 0 correctchoice

d. e

e. ∞

By l'Hopital's Rule: $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

2. The series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ is

a. Divergent by the n^{th} Term Divergence Test

b. Divergent by the Integral Test correctchoice

c. Convergent by the Integral Test

d. Divergent by the Ratio Test

e. Convergent by the Ratio Test

n^{th} Term Divergence Test: $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$ Test Fails.

Integral Test: $\frac{\ln n}{n}$ is decreasing and $\int_1^\infty \frac{\ln n}{n}\, dn = \int_0^\infty u\, du = \left[\frac{u^2}{2} \right]_0^\infty = \infty$

So $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges.

Ratio Test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\ln(n+1)n}{(n+1)\ln(n)} \right| = 1$ Test Fails.

3. A convergent sequence is recursively defined by $a_1 = 1$ and $a_{n+1} = \frac{3-a_n}{2+a_n}$. Find $\lim_{n\to\infty} a_n$.

a. $\frac{19}{24}$

b. $\frac{\sqrt{21}-3}{2}$ correctchoice

c. $\frac{\pi}{4}$

d. $\frac{3}{2}$

e. $\frac{2}{3}$

Let $L = \lim_{n\to\infty} a_n$. Then the limit of the recurrsion relation says $L = \frac{3-L}{2+L}$. Thus

$2L + L^2 = 3 - L$ or $L^2 + 3L - 3 = 0$ or $L = \frac{-3 \pm \sqrt{9-4(-3)}}{2} = \frac{-3 \pm \sqrt{21}}{2}$

4. The sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^{n+1}}{4^n}$ is:

a. nonexistant since its partial sums oscillate

b. $-\frac{3}{4}$

c. $\frac{9}{7}$ correctchoice

d. nonexistant by the n^{th} Term Divergence Test

e. $\frac{3}{16}$

This is a geometric series with ratio $r = -\frac{3}{4}$ and first term $a = \frac{(-1)^2 3^2}{4^1} = \frac{9}{4}$.

So $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}3^{n+1}}{4^n} = \frac{\frac{9}{4}}{1 - -\frac{3}{4}} = \frac{9}{4+3} = \frac{9}{7}$

5. Compute $\sum_{k=2}^{\infty} \frac{3}{(n-1)(n+1)}$. (HINTS: partial fractions,telescoping sum)

a. $\frac{9}{4}$ correctchoice

b. $\frac{3}{2}$

c. 1

d. $\frac{1}{2}$

e. ∞

Partial Fractions: $\frac{3}{(n-1)(n+1)} = \frac{A}{n-1} + \frac{B}{n+1}$ $3 = A(n+1) + B(n-1)$

$n = 1:$ $3 = A(2)$ $A = \frac{3}{2}$ $n = -1:$ $3 = B(-2)$ $B = -\frac{3}{2}$ So

$\frac{3}{(n-1)(n+1)} = \frac{3}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$ and $\sum_{k=2}^{\infty} \frac{3}{(n-1)(n+1)} = \frac{3}{2}\sum_{k=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$

The partial sum is

$$S_k = \frac{3}{2}\sum_{k=2}^{k}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$
$$= \frac{3}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + + \left(\frac{1}{k-2} - \frac{1}{k}\right) + \left(\frac{1}{k-1} - \frac{1}{k+1}\right.\right.$$
$$= \frac{3}{2}\left[\frac{1}{1} + \frac{1}{2} - \frac{1}{k} - \frac{1}{k+1}\right]$$

Thus $\sum_{k=2}^{\infty} \frac{3}{(n-1)(n+1)} = \lim_{n\to\infty} S_k = \frac{3}{2}\left[\frac{1}{1} + \frac{1}{2}\right] = \frac{9}{4}$

6. The series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{3^n n^2}{n!}$ is

a. Absolutely convergent correctchoice

b. Conditionally convergent

c. Divergent

Ratio Test: $a_n = (-1)^n \dfrac{3^n n^2}{n!} \qquad a_{n+1} = (-1)^{n+1} \dfrac{3^{n+1}(n+1)^2}{(n+1)!}$

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(n+1)^2}{(n+1)!}\frac{n!}{3^n n^2}\right| = \lim_{n\to\infty}\left|\frac{3}{n+1}\left(\frac{n+1}{n}\right)^2\right| = 0$$

So the related absolute series converges and hence the original series is absolutely convergent.

7. The series $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{(n+3)2^{2n}}{3^{n+100}}$ is

a. Absolutely convergent

b. Conditionally convergent

c. Divergent correctchoice

n^{th} Term Divergence Test: $\displaystyle\lim_{n\to\infty}\frac{(n+3)2^{2n}}{3^{n+100}} = \frac{1}{3^{100}}\lim_{n\to\infty}(n+3)\left(\frac{4}{3}\right)^n = \infty$ OR

Ratio Test: $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+4)2^{2(n+1)}}{3^{n+101}}\frac{3^{n+100}}{(n+3)2^{2n}}\right| = \lim_{n\to\infty}\frac{4}{3}\frac{(n+4)}{(n+3)} = \frac{4}{3} > 1$

In either case the series is divergent.

8. Given that $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$, compute

$$\lim_{x\to 0}\frac{\ln(1+2x) - 2x + 2x^2}{(2x)^3}$$

a. 0

b. $\frac{8}{3}$

c. $\frac{4}{3}$

d. $\frac{1}{3}$ correctchoice

e. ∞

$$\ln(1+2x) = 2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \frac{16x^4}{4} + \cdots$$

$$\lim_{x\to 0}\frac{\ln(1+2x) - 2x + 2x^2}{(2x)^3} = \lim_{x\to 0}\frac{\left(2x - \frac{4x^2}{2} + \frac{8x^3}{3} - \frac{16x^4}{4} + \cdots\right) - 2x + 2x^2}{(2x)^3}$$

$$= \lim_{x\to 0}\frac{\left(\frac{8x^3}{3} - \frac{16x^4}{4} + \cdots\right)}{8x^3} = \lim_{x\to 0}\frac{\left(\frac{8}{3} - \frac{16x}{4} + \cdots\right)}{8} = \frac{1}{3}$$

9. For what values of x does the series $\sum_{n=0}^{\infty} \frac{1}{(x-3)^n}$ converge?

a. $x > 4$ only
b. $x > 3$ only
c. $0 < x < 3$ only
d. $x < 1$ or $x > 3$ only
e. $x < 2$ or $x > 4$ only   correctchoice

This is a geometric series with ratio $r = \frac{1}{x-3}$. It converges when $|r| < 1$, or $\left|\frac{1}{x-3}\right| < 1$ or $1 < |x-3|$. This means $x - 3 < -1$ or $x - 3 > 1$. Equivalently, $x < 2$ or $x > 4$.

10. In the Maclaurin series for $\frac{\sin(x^2)}{x}$ the coefficient of x^9 is

a. 1
b. $\frac{1}{3!}$
c. $\frac{1}{5!}$ correctchoice
d. $\frac{1}{7!}$
e. $\frac{1}{9!}$

Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ we have
$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$ and $\frac{\sin(x^2)}{x} = x - \frac{x^5}{3!} + \frac{x^9}{5!} - \frac{x^{13}}{7!} + \cdots$.
So the coefficient of x^9 is $\frac{1}{5!}$.

11. (25 points) You are given: $\cos(x^2) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n}}{(2n)!} = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots$.

a. (5 pt) If $f(x) = \cos(x^2)$, find $f^{(6)}(0)$.

The term with an x^6 is $\frac{f^{(6)}(0)}{6!}x^6 = 0$. So: $f^{(6)}(0) = 0$.

b. (5 pt) If $f(x) = \cos(x^2)$, find $f^{(16)}(0)$.

The term with an x^{16} is $\frac{f^{(16)}(0)}{16!}x^{16} = \frac{x^{16}}{8!}$.

So: $f^{(16)}(0) = \frac{16!}{8!} = 16 \bullet 15 \bullet 14 \bullet 13 \bullet 12 \bullet 11 \bullet 10 \bullet 9 = 518918400$.

c. (10 pt) Use the **quartic** (degree 4) Taylor polynomial approximation about $x = 0$ for $\cos(x^2)$ to estimate $\int_0^{0.1} \cos(x^2)\, dx$.

The quartic Taylor polynomial approximation is $\cos(x^2) \approx 1 - \frac{x^4}{2!}$. So:

$$\int_0^{0.1} \cos(x^2)\, dx \approx \int_0^{0.1} 1 - \frac{x^4}{2}\, dx = \left[x - \frac{x^5}{10} \right]_0^{0.1} = .1 - \frac{.1^5}{10} = .099999$$

d. (5 pt) Your result in (c) is equal to $\int_0^{0.1} \cos(x^2)\, dx$ to within ± how much? Why?

Since $\cos(x^2)$ and $\int_0^{0.1} \cos(x^2)\, dx$ are alternating decreasing series, the error is at most the next term:

$$\int_0^{0.1} \frac{x^8}{4!}\, dx = \left[\frac{x^9}{9 \bullet 4!} \right]_0^{0.1} = \frac{(.1)^9}{9 \bullet 24} = 4.6 \times 10^{-12}$$

12. (15 points) Find the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(x-6)^n}{2^n\sqrt{n}}$.

Be sure to identify each of the following and give reasons:

(1 pt) Center of Convergence: $a = \underline{\quad 6 \quad}$

$$a_n = \frac{(x-6)^n}{2^n\sqrt{n}} \qquad a_{n+1} = \frac{(x-6)^{n+1}}{2^{n+1}\sqrt{n+1}}$$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-6)^{n+1}}{2^{n+1}\sqrt{n+1}}\,\frac{2^n\sqrt{n}}{(x-6)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-6)}{2}\sqrt{\frac{n}{n+1}}\right| = \frac{|x-6|}{2}$$

The series converges if $\frac{|x-6|}{2} < 1$ or $|x-6| < 2$

Radius of Convergence: $R = \underline{\quad 2 \quad}$ (5 pt)

(1 pt) Right Endpoint: $x = \underline{\quad 6+2 = 8 \quad}$

The series $\sum_{n=1}^{\infty} \frac{(8-6)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because it is s p-series with $p = \frac{1}{2} < 1.$

At the Right Endpoint the Series { Converges / **[Diverges]** } (circle one) (3 pt)

(1 pt) Left Endpoint: $x = \underline{\quad 6-2 = 4 \quad}$

The series $\sum_{n=1}^{\infty} \frac{(4-6)^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges because it is an alternating decreasing series and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$

At the Left Endpoint the Series { **[Converges]** / Diverges } (circle one) (3 pt)

(1 pt) Interval of Convergence: $\underline{\quad 4 \le x < 8 \quad}$ or $\underline{\quad [4,8) \quad}$