MATH 172 EXAM 1 Fall 1999

Section 502 Solutions P. Yasskin

Multiple Choice: (5 points each)

1. Compute: $\int \sqrt{x}\left(x^2 - \frac{1}{x}\right) dx$

a. $\frac{2x^{7/2}}{7} + \frac{2x^{-3/2}}{3} + C$

b. $\frac{2x^{7/2}}{7} - 2x^{1/2} + C$ correctchoice

c. $\frac{2x^{37/2}}{3} + \frac{2x^{-3/2}}{3} + C$

d. $\sqrt{x}\left(\frac{x^3}{3} - \ln x\right) + \frac{2x^{3/2}}{3}(x^2 - \ln x) + C$

e. $\frac{2x^{3/2}}{3}\left(\frac{x^3}{3} - \ln x\right) + C$

$$\int \sqrt{x}\left(x^2 - \frac{1}{x}\right) dx = \int (x^{5/2} - x^{-1/2})\, dx = \frac{2x^{7/2}}{7} - \frac{2x^{1/2}}{1} + C$$

2. Evaluate $\int_0^2 \sqrt{4 - x^2} + 1 \; dx$ by interpretating it as an area.

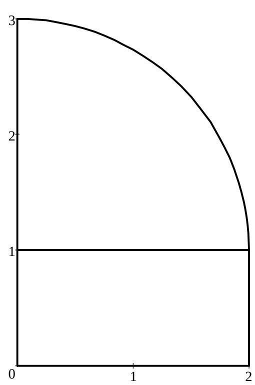
a. $2 + 2\pi$

b. $1 + \pi$

c. $1 + 2\pi$

d. $\pi - 1$

e. $2 + \pi$ correctchoice



Quarter circle of radius 2 on top of rectangle of width 2 and height 1.

$A = \frac{1}{4}\pi(2)^2 + 2 \bullet 1 = \pi + 2$

3. Find the area between the parabola $y = x^2$ and the line $y = 2x + 3$.

- **a.** $-\frac{16}{3}$
- **b.** $\frac{16}{3}$
- **c.** $\frac{32}{3}$ correctchoice
- **d.** 9
- **e.** $\frac{88}{3}$

$$x^2 = 2x + 3 \quad \Rightarrow \quad x^2 - 2x - 3 = 0 \quad \Rightarrow \quad (x+1)(x-3) = 0 \quad \Rightarrow \quad x = -1, 3$$

$$A = \int_{-1}^{3} (2x + 3 - x^2)\,dx = \left[x^2 + 3x - \frac{x^3}{3} \right]_{-1}^{3} = [9 + 9 - 9] - \left[1 - 3 + \frac{1}{3} \right] = 9 + 2 - \frac{1}{3} = \frac{32}{3}$$

4. Compute $\int_1^4 \frac{\ln x}{2\sqrt{x}}\,dx$

- **a.** $2\ln 4 - 1$
- **b.** $2\ln 4 - 2$ correctchoice
- **c.** $2\ln 4 - 4$
- **d.** $2\ln 4 - 6$
- **e.** $\ln 2 - 4$

Integrate by parts with

$$u = \ln x \qquad dv = \frac{1}{2\sqrt{x}}\,dx$$

$$du = \frac{1}{x}\,dx \qquad v = \sqrt{x}$$

$$\int_1^4 \frac{\ln x}{2\sqrt{x}}\,dx = \sqrt{x}\ln x - \int \frac{1}{\sqrt{x}}\,dx = \left[\sqrt{x}\ln x - 2\sqrt{x} \right]_1^4 = \left[\sqrt{4}\ln 4 - 2\sqrt{4} \right] - \left[\sqrt{1}\ln 1 - 2\sqrt{1} \right]$$

$$= 2\ln 4 - 4 + 2 = 2\ln 4 - 2$$

5. The mass density of a 3 cm bar is $\rho = 1 + x^2 \ \frac{\text{gm}}{\text{cm}}$ for $0 \le x \le 3$. Find the total mass of the bar.

- **a.** 4 gm
- **b.** 10 gm
- **c.** 12 gm correctchoice
- **d.** 18 gm
- **e.** 30 gm

$$M = \int_0^3 \rho(x)\,dx = \int_0^3 (1 + x^2)\,dx = \left[x + \frac{x^3}{3} \right]_0^3 = 3 + 9 = 12$$

6. The mass density of a 3 cm bar is $\rho = 1 + x^2 \ \frac{\text{gm}}{\text{cm}}$ for $0 \le x \le 3$. Find the x-coordinate of the center of mass of the bar.

a. $\frac{3}{2}$

b. 2

c. $\frac{7}{3}$

d. $\frac{33}{16}$ correctchoice

e. $\frac{99}{4}$

$$M_1 = \int_0^3 x\rho(x)\,dx = \int_0^3 (x + x^3)\,dx = \left[\frac{x^2}{2} + \frac{x^4}{4}\right]_0^3 = \frac{9}{2} + \frac{81}{4} = \frac{99}{4}$$

$$\bar{x} = \frac{M_1}{M} = \frac{99}{4 \bullet 12} = \frac{33}{16}$$

7. Compute $\int_0^{\pi/2} \sin^4\theta \cos\theta \ d\theta$

a. $-\frac{1}{3}$

b. $-\frac{1}{5}$

c. $\frac{1}{6}$

d. $\frac{1}{5}$ correctchoice

e. $\frac{1}{3}$

$$\int_0^{\pi/2} \sin^4\theta \cos\theta \ d\theta = \frac{\sin^5\theta}{5}\Big|_0^{\pi/2} = \frac{1}{5}$$

8. Compute $\int_0^{\pi/4} \tan^3\theta \sec^2\theta \ d\theta$

a. $-\frac{1}{4}$

b. $\frac{1}{4}$ correctchoice

c. $-\frac{1}{3}$

d. $\frac{1}{3}$

e. $-\frac{1}{2}$

$$\int_0^{\pi/4} \tan^3\theta \sec^2\theta \ d\theta = \frac{\tan^4\theta}{4}\Big|_0^{\pi/4} = \frac{1}{4}$$

9. Compute: $\int_0^2 x\sqrt{4-x^2}\,dx$

a. $\frac{2\sqrt{2}}{3}$

b. $\frac{8}{3}$ correctchoice

c. 24

d. $\frac{32}{3}$

e. 6

$$u = 4 - x^2 \qquad du = -2x\,dx \qquad -\frac{1}{2}\,du = x\,dx$$

$$\int_0^2 x\sqrt{4-x^2}\,dx = -\frac{1}{2}\int_{x=0}^{2}\sqrt{u}\,du = -\frac{1}{2}\left[\frac{2u^{3/2}}{3}\right] = -\frac{1}{3}\left[(4-x^2)^{3/2}\right]_0^2 = 0 + \frac{1}{3}(4)^{3/2} = \frac{8}{3}$$

10. (15 points) Compute $\int_0^1 (t^2 - t)\,e^{2t}\,dt$

$$\int_0^1 (t^2-t)\,e^{2t}\,dt = (t^2-t)\frac{1}{2}e^{2t} - \frac{1}{2}\int(2t-1)\,e^{2t}\,dt$$

$u = t^2 - t \qquad dv = e^{2t}\,dt$

$du = (2t-1)\,dt \qquad v = \frac{1}{2}e^{2t}$

$$= (t^2-t)\frac{1}{2}e^{2t} - \frac{1}{2}\left((2t-1)\frac{1}{2}e^{2t} - \frac{1}{2}\int 2\,e^{2t}\,dt\right)$$

$u = 2t - 1 \qquad dv = e^{2t}\,dt$

$du = 2\,dt \qquad v = \frac{1}{2}e^{2t}$

$$= (t^2-t)\frac{1}{2}e^{2t} - \frac{1}{2}\left((2t-1)\frac{1}{2}e^{2t} - \frac{1}{2}e^{2t}\right)$$

$$= \left[\frac{1}{2}(t^2-t)\,e^{2t} - \frac{1}{4}(2t-1)\,e^{2t} + \frac{1}{4}e^{2t}\right]_0^1$$

$$= \left[0 - \frac{1}{4}(1)\,e^2 + \frac{1}{4}e^2\right] - \left[0 - \frac{1}{4}(-1)\,e^0 + \frac{1}{4}e^0\right]$$

$$= -\frac{1}{2}$$

11. (15 points) Compute $\int_1^2 \frac{\sqrt{x^2-1}}{x}\, dx$

You must evaluate any inverse trig functions in the answer.

$x = \sec\theta \qquad dx = \sec\theta\tan\theta\, d\theta \qquad \sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\, dx = \int_{x=1}^2 \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\, d\theta = \int_{x=1}^2 \tan^2\theta\, d\theta$$
$$= \int_{x=1}^2 (\sec^2\theta - 1)\, d\theta = \Big[\tan\theta - \theta\Big]_{x=1}^2 = \Big[\sqrt{x^2-1} - \operatorname{arcsec} x\Big]_{x=1}^2$$
$$= \Big[\sqrt{4-1} - \operatorname{arcsec} 2\Big] - \Big[\sqrt{1-1} - \operatorname{arcsec} 1\Big] = \sqrt{3} - \frac{\pi}{3}$$

because $\operatorname{arcsec} 2 = \frac{\pi}{3}$ and $\operatorname{arcsec} 1 = 0$.

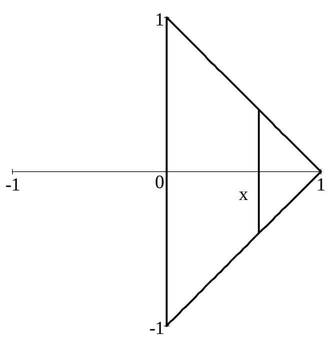
12. (15 points) Find the arc length of the parametric curve $x = \frac{1}{2}t^6,\ \ y = t^4$ between $t = 0$ and $t = 1$.

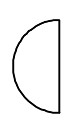
HINT: Factor the quantity in the square root.

$\frac{dx}{dt} = 3t^5 \qquad \frac{dy}{dt} = 4t^3$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^1 \sqrt{9t^{10} + 16t^6}\, dt = \int_0^1 t^3\sqrt{9t^4+16}\, dt$$
$$= \frac{1}{36}\int_{t=0}^1 \sqrt{u}\, du \qquad \text{where} \qquad u = 9t^4 + 16 \qquad du = 36t^3\, dt$$
$$= \frac{1}{36}\frac{2u^{3/2}}{3}\Big|_{t=0}^1 = \frac{1}{54}(9t^4+16)^{3/2}\Big|_{t=0}^1 = \frac{1}{54}(25)^{3/2} - \frac{1}{54}(16)^{3/2} = \frac{125-64}{54} = \frac{61}{54}$$

13. (10 points) Find the volume of the solid whose base is the triangle with vertices $(0,-1)$, $(1,0)$ and $(0,1)$ and whose crosssections perpendicular to the x-axis are semicircles.





The top of the triangle is

$y = 1 - x$

So the radius of the semicircle is

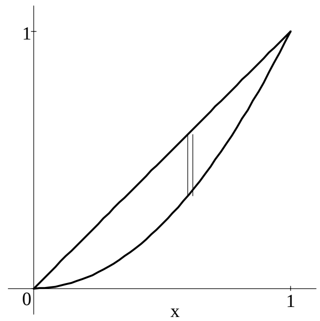
$r - 1 - x$

So the crosssectional area is $A(x) = \frac{1}{2}\pi r^2 = \frac{\pi}{2}(1-x)^2$ and the volume is

$$V = \int_0^1 A(x)\, dx = \int_0^1 \frac{\pi}{2}(1-x)^2\, dx = -\left[\frac{\pi}{2}\frac{(1-x)^3}{3}\right]_0^1 = 0 + \left[\frac{\pi}{2}\frac{(1)^3}{3}\right] = \frac{\pi}{6}$$

14. (10 points) The area between the parabola $y = x^2$ and the line $y = x$ is rotated about the indicated axis. Find the volume of the solid swept out.

a. x-axis



This is an x-integral. Use a vertical rectangle.

Rotating about the x-axis, gives washers.

upper radius $= x$ lower radius $= x^2$

$$V = \int_0^1 \pi(x)^2 - \pi(x^2)^2\,dx = \pi\int_0^1 x^2 - x^4\,dx$$

$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left[\frac{1}{3} - \frac{1}{5}\right] = \pi\frac{5-3}{15} = \frac{2\pi}{15}$$

b. y-axis

This is again an x-integral. Again use a vertical rectangle.

Rotating about the y-axis, gives cylindrical shells.

radius $= x$ height $= x - x^2$

$$V = \int_0^1 2\pi x(x - x^2)\,dx = 2\pi\int_0^1 x^2 - x^3\,dx$$

$$= 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 2\pi\left[\frac{1}{3} - \frac{1}{4}\right] = 2\pi\frac{4-3}{12} = \frac{\pi}{6}$$