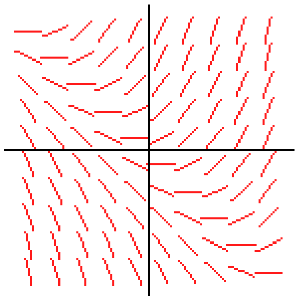
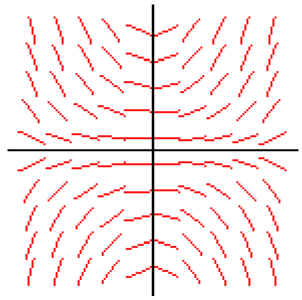
MATH 172 EXAM 3 Fall 1999

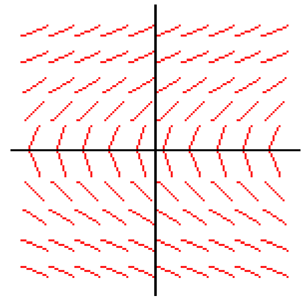
Section 502 Solutions P. Yasskin

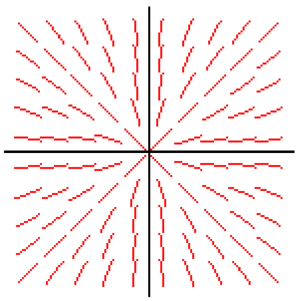
Multiple Choice: (8 points each)

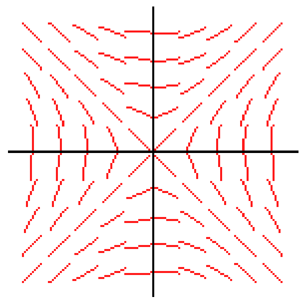
1. Which of the following is the direction field of the differential equation $\dfrac{dy}{dx} = xy$?

a. 

b. 

c. 

d. 

e. 

The slope is $m = xy$ which is 0 on both axis, positive in quadrants I and III, and negative in quadrants II and IV. This only happens in plot (b).

2. The limit $\lim\limits_{n\to\infty} \dfrac{4(2)^n}{(5)^n}$

a. converges to 0. correctchoice

b. converges to $\dfrac{4}{1-\frac{2}{5}}$.

c. converges to 4.

d. converges to $\dfrac{\frac{8}{5}}{1-\frac{2}{5}}$.

e. diverges.

$$\lim_{n\to\infty} \frac{4(2)^n}{(5)^n} = 4\lim_{n\to\infty}\left(\frac{2}{5}\right)^n = 0$$

This is NOT a geometric series. It is not being added up.

3. $\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^{3n^2} =$

a. e^{-6}
b. e^{6}
c. 1
d. e
e. e^{-3} correctchoice

$\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^{3n^2} = e^L$ where

$$L = \lim_{n\to\infty}\ln\left(1-\frac{1}{n^2}\right)^{3n^2} = \lim_{n\to\infty}3n^2\ln\left(1-\frac{1}{n^2}\right) = \lim_{n\to\infty}3\frac{\ln\left(1-\frac{1}{n^2}\right)}{\frac{1}{n^2}}$$

$$= 3\lim_{n\to\infty}\frac{\frac{1}{1-\frac{1}{n^2}}\left(\frac{2}{n^3}\right)}{\frac{-2}{n^3}} = 3\lim_{n\to\infty}\frac{-1}{1-\frac{1}{n^2}} = -3$$

So $\lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^{3n^2} = e^{-3}$

4. The series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n}{3^n}$

a. diverges by the n^{th}-Term Test.
b. converges to 2
c. converges to $\frac{2}{5}$ correctchoice
d. converges to $-\frac{3}{5}$
e. diverges by the Alternating Series Test

Geometric series with $a = \frac{(-1)^{1+1}2^1}{3^1} = \frac{2}{3}$ and $r = \frac{-2}{3}$.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^n}{3^n} = \frac{\frac{2}{3}}{1-\frac{-2}{3}} = \frac{2}{3+2} = \frac{2}{5}$$

5. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{n}-\frac{n+2}{n+1}\right)=$

- **a.** 0
- **b.** 1 correctchoice
- **c.** 2
- **d.** 3
- **e.** ∞

$$\begin{aligned} S_k &= \sum_{n=1}^{k}\left(\frac{n+1}{n}-\frac{n+2}{n+1}\right) \\ &= \left(\frac{2}{1}-\frac{3}{2}\right)+\left(\frac{3}{2}-\frac{4}{3}\right)+\cdots\left(\frac{k}{k-1}-\frac{k+1}{k}\right)+\left(\frac{k+1}{k}-\frac{k+2}{k+1}\right) \\ &= 2-\frac{k+2}{k+1} \end{aligned}$$

$$S=\lim_{k\to\infty}S_k=\lim_{k\to\infty}\left(2-\frac{k+2}{k+1}\right)=2-1=1$$

6. The series $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{2^n}$

- **a.** converges by the Integral Test.
- **b.** diverges by the Integral Test.
- **c.** converges by the Ratio Test.
- **d.** diverges by the Ratio Test. correctchoice
- **e.** converges by Comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$.

Ratio Test: $a_n=\dfrac{n!}{2^n}$ $\quad a_{n+1}=\dfrac{(n+1)!}{2^{n+1}}$

$$L=\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to\infty}\frac{(n+1)!}{2^{n+1}}\frac{2^n}{n!}=\lim_{n\to\infty}\frac{(n+1)}{2}=\infty>1$$ Divergent

7. The series $\sum_{n=2}^{\infty} \frac{2}{n^3 + \sqrt{n}}$

- **a**. diverges by the n^{th}-Term Test.
- **b**. converges by the Ratio Test.
- **c**. diverges by the Ratio Test.
- **d**. converges by Comparison to $\sum_{n=2}^{\infty} \frac{2}{n^3}$ correctchoice
- **e**. diverges by Comparison to $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}}$

The n^{th}-Term Test and the Ratio Test fail.

For large n, we have $n^3 > \sqrt{n}$. So we compare to $\sum_{n=2}^{\infty} \frac{2}{n^3}$ which is a p-series with $p = 3 > 1$ and so is convergent. Further, $n^3 + \sqrt{n} > n^3$. So $\frac{2}{n^3 + \sqrt{n}} < \frac{2}{n^3}$ and $\sum_{n=2}^{\infty} \frac{2}{n^3 + \sqrt{n}} < \sum_{n=2}^{\infty} \frac{2}{n^3}$. Therefore $\sum_{n=2}^{\infty} \frac{2}{n^3 + \sqrt{n}}$ is also convergent.

8. The series $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n+1}}$

- **a**. converges by the Integral Test.
- **b**. diverges by the Integral Test. correctchoice
- **c**. converges by the Ratio Test.
- **d**. diverges by the Ratio Test.
- **e**. diverges by the n^{th}-Term Test.

The n^{th}-Term Test and the Ratio Test fail.
Integral Test:
$\int_2^{\infty} \frac{1}{\sqrt{n+1}}\, dn = \int_2^{\infty} (n+1)^{-1/2}\, dn = 2(n+1)^{1/2}\Big|_2^{\infty} = \infty$ Series is divergent.

9. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is

a. Absolutely Convergent
b. Conditionally Convergent correctchoice
c. Convergent for Even n, Divergent for Odd n
d. Convergent for Odd n, Divergent for Even n
e. Divergent

The related absolute series is $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ which is convergent from the previous problem.
The original series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is alternating from the $(-1)^n$, decreasing since $\frac{1}{\sqrt{n+1}}$ gets smaller as n gets bigger and $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$. So it is convergent by the Alternating Series Test. Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ is Conditionally Convergent.

10. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ is

a. Absolutely Convergent correctchoice
b. Conditionally Convergent
c. Convergent for Even n, Divergent for Odd n
d. Convergent for Odd n, Divergent for Even n
e. Divergent

The related absolute series is $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ which is a p-series with $p = \frac{3}{2} > 1$ and hence convergent.
Therefore the original series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ is Absolutely Convergent.

11. (15 points) Solve the differential equation $\frac{1}{x}\frac{dy}{dx} + 3xy = e^{-x^3}$ with $y(0) = 2$

We write the equation in standard linear form:

$$\frac{dy}{dx} + 3x^2 y = xe^{-x^3}$$

We identify $P = 3x^2$ and compute the integrating factor:

$$I = e^{\int P\,dx} = e^{\int 3x^2\,dx} = e^{x^3}$$

We multiply thru by e^{x^3} and identify the left as the derivative of a product:

$$\frac{d}{dx}\left(e^{x^3}y\right) = e^{x^3}\frac{dy}{dx} + e^{x^3}3x^2 y = x$$

We integrate:

$$e^{x^3}y = \int x\,dx = \frac{1}{2}x^2 + C$$

We use the initial condition $y = 2$ when $x = 0$:

$$e^0 2 = \frac{1}{2}0^2 + C \qquad \Rightarrow \qquad C = 2$$

Therefore

$$y = \frac{1}{2}x^2 e^{-x^3} + 2e^{-x^3}$$

12. (15 points) A tank contains 5000 gal of water. Initially, there are 10 lb of salt dissolved in the water. Salt water containing 0.03 lb of salt per gal is added to the tank at the rate of 2 gal per hour. The solution is kept mixed and is drained at the rate of 2 gal per hour. Let $S(t)$ be the amount of salt in the tank at time t.

a. State the differential equation and the initial condition satisfied by $S(t)$.

$$\frac{dS}{dt} = \underbrace{.03\frac{\text{lb}}{\text{gal}} \bullet 2\frac{\text{gal}}{\text{hour}}}_{\text{in}} - \underbrace{\frac{S\text{ lb}}{5000\text{ gal}} \bullet 2\frac{\text{gal}}{\text{hour}}}_{\text{out}} = .06 - \frac{1}{2500}S$$

$$S(0) = 10$$

b. Solve this initial value problem.

$$\frac{dS}{dt} + \frac{1}{2500}S = .06 \qquad \Rightarrow \qquad I = \exp\left(\int \frac{1}{2500}dt\right) = e^{t/2500}$$

$$\Rightarrow \qquad \frac{d}{dt}\left(e^{t/2500}S\right) = .06e^{t/2500}$$

$$\Rightarrow \qquad e^{t/2500}S = \int .06e^{t/2500}dt = .06 \bullet 2500e^{t/2500} + C = 150e^{t/2500} + C$$

$$\Rightarrow \qquad S = 150 + Ce^{-t/2500}$$

Initial condition: $10 = 150 + C \qquad \Rightarrow \qquad C = -140$

Solution: $S = 150 - 140e^{-t/2500}$

c. At large times, what is the asymptotic amount of salt in the tank.

$$\lim_{t\to\infty} S(t) = \lim_{t\to\infty} 150 - 140e^{-t/2500} = 150 \text{ lb}$$