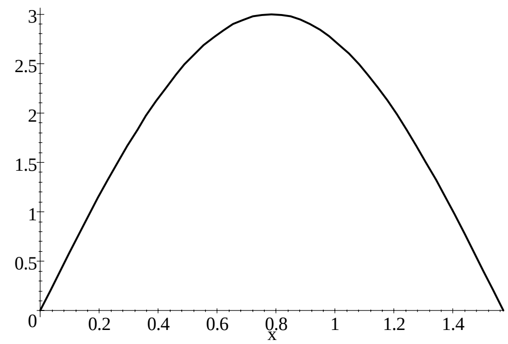
MATH 172 FINAL EXAM Fall 1999

Section 502 Solutions P. Yasskin

Multiple Choice: (5 points each)

1. Find the area below $y = 3\sin(2x)$ above the x-axis for $0 \le x \le \frac{\pi}{2}$.



a. $\frac{1}{2}$

b. $\frac{3}{2}$

c. $\frac{\pi}{2}$

d. $\frac{3\pi}{2}$

e. 3 correctchoice

$$A = \int_0^{\pi/2} 3\sin(2x)\,dx = \left[-\frac{3\cos(2x)}{2}\right]_0^{\pi/2} = -\frac{3\cos\pi}{2} + \frac{3\cos 0}{2} = -\frac{3(-1)}{2} + \frac{3}{2} = 3$$

2. The region below $y = 3\sin(2x)$ above the x-axis for $0 \le x \le \frac{\pi}{2}$ is rotated about the y-axis. (See the figure in problem 1.) Which formula will give the volume of the solid of revolution?

a. $A = \int_0^{\pi/2} x^2 \sin(2x)\,dx$

b. $A = \int_0^{\pi/2} 3x\sin(2x)\,dx$

c. $A = \int_0^{\pi/2} 6\pi x\sin(2x)\,dx$ correctchoice

d. $A = \int_0^{\pi/2} 9\pi \sin^2(2x)\,dx$

e. $A = \int_0^{\pi/2} 18\pi \sin^2(2x)\,dx$

x-integral, Cylinders: Radius is $r = x$. Height is $h = y = 3\sin(2x)$

$$V = \int 2\pi rh\,dx = \int_0^{\pi/2} 2\pi x 3\sin(2x)\,dx = \int_0^{\pi/2} 6\pi x\sin(2x)\,dx$$

3. A 1 m bar has linear mass density $\rho = \frac{1}{1+x^2}\ \frac{\text{kg}}{\text{m}}$ where x is measured from one end. Find the total mass.

a. $M = \frac{\pi}{4}$ kg   correctchoice
b. $M = \frac{\pi}{2}$ kg
c. $M = \frac{1}{2}$ kg
d. $M = 45$ kg
e. $M = 90$ kg

$$M = \int \rho\, dx = \int_0^1 \frac{1}{1+x^2}\, dx = \Big[\arctan x\Big]_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4}$$

4. A 1 m bar has linear mass density $\rho = \frac{1}{1+x^2}\ \frac{\text{kg}}{\text{m}}$ where x is measured from one end. Find the center of mass.

a. $\bar{x} = \frac{\ln 2}{90}$ m
b. $\bar{x} = \frac{\ln 2}{2}$ m
c. $\bar{x} = \frac{2\ln 2}{\pi}$ m   correctchoice
d. $\bar{x} = \frac{\ln 2}{2\pi}$ m
e. $\bar{x} = \frac{1}{2}$ m

$$M_1 = \int x\rho\, dx = \int_0^1 \frac{x}{1+x^2}\, dx = \left[\frac{1}{2}\ln(1+x^2)\right]_0^1 = \frac{1}{2}\ln 2 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 2$$
$$\bar{x} = \frac{M_1}{M} = \frac{\ln 2}{2}\frac{4}{\pi} = \frac{2\ln 2}{\pi}$$

5. Compute $\int_{-\pi/2}^{\pi/2} \sin^6\theta \cos\theta\, d\theta$

a. $-\frac{2}{7}$
b. $-\frac{1}{7}$
c. 0
d. $\frac{1}{7}$
e. $\frac{2}{7}$ correctchoice

$u = \sin\theta \qquad du = \cos\theta\, d\theta$

When $\theta = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2} = 1$.   When $\theta = -\frac{\pi}{2}$, $u = \sin\left(-\frac{\pi}{2}\right) = -1$.

$$\int_{-\pi/2}^{\pi/2} \sin^6\theta \cos\theta\, d\theta = \int_{-1}^{1} u^6\, du = \left[\frac{u^7}{7}\right]_{-1}^{1} = \frac{1}{7} - \frac{-1}{7} = \frac{2}{7}$$

6. The curve $y = x^3$ for $0 \le x \le 3$ is rotated about the x-axis. Which formula will give the area of the surface of revolution?

a. $A = \int_0^3 2\pi x \sqrt{1 + 9x^4}\, dx$

b. $A = \int_0^3 2\pi x^3 \sqrt{1 + 9x^4}\, dx$ correctchoice

c. $A = \int_0^3 2\pi x^3\, dx$

d. $A = \int_0^3 2\pi x (3x^2)\, dx$

e. $A = \int_0^3 \pi x \sqrt{1 + 9x^4}\, dx$

x-integral, Radius is $r = y = x^3$ $\quad \dfrac{dy}{dx} = 3x^2$

$$A = \int 2\pi r\, ds = \int_0^3 2\pi x^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 2\pi x^3 \sqrt{1 + 9x^4}\, dx$$

7. Compute $\displaystyle\int_0^2 \frac{2x}{4 - x^2}\, dx$

a. $-\infty$

b. $-\ln 4$

c. $\dfrac{\pi}{4}$

d. $\ln 4$

e. ∞ correctchoice

$u = 4 - x^2$ $\quad du = -2x\, dx$ $\quad x = 0$ at $u = 4$ $\quad x = 2$ at $u = 0$

$$\int_0^2 \frac{2x}{4 - x^2}\, dx = -\int_4^0 \frac{1}{u}\, du = -\Big[\ln u\Big]_4^0 = -[-\infty] + [\ln 4] = \infty$$

8. If it requires 24 J of **work** to stretch a spring from rest to 4 m, how much work will it take to stretch it from 2 m to 6 m?

a. 6 J

b. 12 J

c. 24 J

d. 48 J correctchoice

e. 96 J

$$W = \int_0^4 kx\, dx = \left[k\frac{x^2}{2}\right]_0^4 = 8k = 24 \quad \Rightarrow \quad k = 3$$

$$W = \int_2^6 3x\, dx = \left[3\frac{x^2}{2}\right]_2^6 = \frac{3}{2}(36 - 4) = 48$$

9. Which term is incorrect in the following partial fraction expansion?

$$\frac{x^3 - 2x + 3}{(x-2)^2(x-3)(x^2+4)} = \underbrace{\frac{A}{x-2}}_{\text{a.}} + \underbrace{\frac{Bx+C}{(x-2)^2}}_{\text{b. correctchoice}} + \underbrace{\frac{D}{x-3}}_{\text{c.}} + \underbrace{\frac{Ex+F}{x^2+4}}_{\text{d.}}$$

e. They are all correct.

Term (b) is incorrect. It should be $\frac{C}{(x-2)^2}$.

If the denominator is linear to a power, the numerator is a constant.

10. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{n^2}{3^n}(x-2)^n$.

a. 0

b. $\frac{1}{3}$

c. 3 correctchoice

d. 9

e. ∞

Ratio Test:

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2|x-2|^{n+1}}{3^{n+1}} \frac{3^n}{n^2|x-2|^n} = \frac{|x-2|}{3} \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = \frac{|x-2|}{3}$$

Converges if $\frac{|x-2|}{3} < 1$ or $|x-2| < 3 = R$

11. (10 points) Compute $\int_0^{\pi/2} 3x\cos(2x)\,dx$

Integration by Parts with
$$u = 3x \qquad dv = \cos(2x)\,dx$$
$$du = 3dx \qquad v = \frac{\sin(2x)}{2}$$

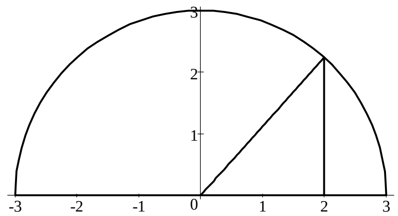
$$\int_0^{\pi/2} 3x\sin(2x)\,dx = \left[3x\frac{\sin(2x)}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} 3\frac{\sin(2x)}{2}\,dx = \left[3x\frac{\sin(2x)}{2} + \frac{3}{2}\frac{\cos(2x)}{2}\right]_0^{\pi/2}$$
$$= \left[3\frac{\pi}{2}\frac{\sin\pi}{2} + \frac{3}{2}\frac{\cos\pi}{2}\right] - \left[0 + \frac{3}{2}\frac{\cos 0}{2}\right] = -\frac{3}{4} - \frac{3}{4} = -\frac{3}{2}$$

12. (10 points) Find the length of the parametric curve given by $x = t^2$, $y = \frac{2}{3}t^3$, $z = \frac{1}{4}t^4$ for $0 \le t \le 2$.

HINT: Factor the quantity in the square root.

$$L = \int ds = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_0^2 \sqrt{(2t)^2 + (2t^2)^2 + (t^3)^2}\,dt$$
$$= \int_0^2 \sqrt{4t^2 + 4t^4 + t^6}\,dt = \int_0^2 t\sqrt{4 + 4t^2 + t^4}\,dt = \int_0^2 t(2+t^2)dt = \left[t^2 + \frac{t^4}{4}\right]_0^2 = 8$$

13. (10 points) Find the volume of the solid whose base is the **semi**-circle $x^2 + y^2 = 9$ for $y \ge 0$ and whose crosssections perpendicular to the x-axis are **squares**.



The side of the square at x is $s = y = \sqrt{9 - x^2}$.
So the area of the square is $A(x) = s^2 = 9 - x^2$.
So the volume is

$$V = \int_{-3}^{3} A(x)\,dx = \int_{-3}^{3} 9 - x^2\,dx = \left[9x - \frac{x^3}{3}\right]_{-3}^{3} = [27 - 9] - [-27 + 9] = 36$$

14. (10 points) Solve the differential equation $\frac{dy}{dx} = 1 + x^2 + y^2 + y^2x^2$ with the initial condition $y(3) = 0$.

Separate variables:

$$\frac{dy}{dx} = (1 + x^2)(1 + y^2) \quad \Rightarrow \quad \int \frac{dy}{1 + y^2} = \int (1 + x^2)\,dx \quad \Rightarrow \quad \arctan y = x + \frac{x^3}{3} + C$$

Use the initial condition:

$$\arctan 0 = 3 + \frac{3^3}{3} + C \quad \Rightarrow \quad C = -12$$

Plug back and solve for y:

$$\arctan y = x + \frac{x^3}{3} - 12 \quad \Rightarrow \quad y = \tan\left(x + \frac{x^3}{3} - 12\right)$$

15. (10 points) Given the series $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$,

a. (5 pts) compute the series for e^{2x}

Substitute $x \Rightarrow 2x$:

$$e^{2x} = 1 + 2x + \frac{1}{2}(2x)^2 + \frac{1}{6}(2x)^3 + \cdots = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

b. (5 pts) and use it to compute $\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2}$.

(2 pts only for l'Hospital's Rule.)

$$\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x^2} = \lim_{x \to 0} \frac{\left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots\right) - 1 - 2x}{x^2} = \lim_{x \to 0} \frac{\left(2x^2 + \frac{4}{3}x^3 + \cdots\right)}{x^2}$$

$$= \lim_{x \to 0}\left(2 + \frac{4}{3}x + \cdots\right) = 2$$