Name________________________________

MATH 172 Honors Exam 1 Spring 2019

Sections 200 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the area between $y = x^2 - 8$ and $y = 2x$.

a. 24

b. $\frac{80}{3}$

c. 36 correct choice

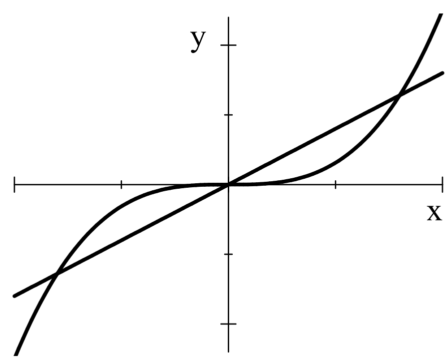
d. $\frac{124}{3}$

e. 48

Solution: The curves intersect when $x^2 - 8 = 2x$ or $0 = x^2 - 2x - 8 = (x+2)(x-4)$ or $x = -2, 4$.

$$A = \int_{-2}^{4} (2x - x^2 + 8)\,dx = \left[x^2 - \frac{x^3}{3} + 8x\right]_{-2}^{4} = \left(16 - \frac{64}{3} + 32\right) - \left(4 - \frac{-8}{3} - 16\right) = 60 - \frac{72}{3} = 36$$

2. Find the area between $y = x^3$ and $y = 16x$.



a. 32

b. 36

c. 48

d. 64

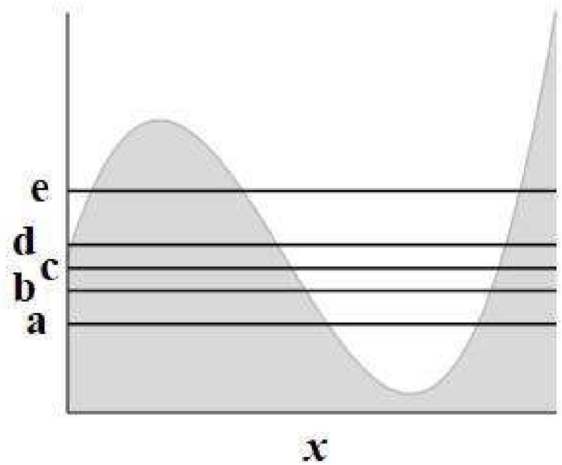
e. 128 correct choice

Solution: The curves intersect when $x^3 = 16x$ or $0 = x^3 - 16x = x(x^2 - 16)$ or $x = 0, \pm 4$.
Since the region is symmetric, we can double the right half:

$$A = 2\int_{0}^{4} 16x - x^3\,dx = 2\left[8x^2 - \frac{x^4}{4}\right]_{0}^{4} = 2(8 \cdot 16 - 64) = 128$$

1-13	/65	15	/10
14	/20	16	/15
		Total	/110

3. Which value is the average of the function?



Solution: (d) is correct.
The empty area below the line must equal the filled area above the line. Don't forget the area above the line on the right. (b) and (c) do not have enough empty area below the line.

4. Compute $\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$.

a. 1 correct choice
b. 2
c. 3
d. 4
e. 6

Solution: Let $u = x^2$. Then $du = 2x\,dx$ and

$$\int_0^{\sqrt{\pi}} x\sin(x^2)\,dx = \frac{1}{2}\int_0^{\pi} \sin u\,du = \left[-\frac{1}{2}\cos u\right]_0^{\pi} = \frac{1}{2} + \frac{1}{2} = 1$$

5. Compute $\int (x^2+1)e^{2x}\,dx$.

a. $\frac{1}{2}(x^2+1)e^{2x} - \frac{1}{4}xe^{2x} + \frac{1}{4}e^{2x} + C$
b. $\frac{1}{2}(x^2+1)e^{2x} - \frac{1}{2}xe^{2x} - \frac{1}{2}e^{2x} + C$
c. $\frac{1}{2}(x^2+1)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$ correct choice
d. $\frac{1}{2}(x^2+1)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}e^{2x} + C$
e. $\frac{1}{2}(x^2+1)e^{2x} - \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

Solution: Use parts with $u = x^2+1 \quad dv = e^{2x}\,dx$, $du = 2x\,dx \quad v = \frac{1}{2}e^{2x}$

$$I = \frac{1}{2}(x^2+1)e^{2x} - \int xe^{2x}\,dx$$

Now use parts with $u = x \quad dv = e^{2x}\,dx$, $du = dx \quad v = \frac{1}{2}e^{2x}$

$$I = \frac{1}{2}(x^2+1)e^{2x} - \left[\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}\,dx\right]$$

$$I = \frac{1}{2}(x^2+1)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$$

6. Find the average value of the function $f(x) = 9 - x^2$ on the interval $[0, 3]$.

a. $\frac{27}{4}$

b. 6 correct choice

c. 5

d. $\frac{9}{2}$

e. 3

Solution: $f_{\text{ave}} = \frac{1}{3}\int_0^3 (9 - x^2)\,dx = \frac{1}{3}\left[9x - \frac{x^3}{3}\right]_0^3 = \frac{1}{3}(27 - 9) = 6$

7. Find the length of the parametric curve $x = t^4$ and $y = \frac{1}{2}t^6$ for $0 \le t \le 1$.

a. $\frac{13}{6}$

b. $\frac{13}{3}$

c. $\frac{13}{2}$

d. $\frac{1}{54}$

e. $\frac{61}{54}$ correct choice

Solution: $\frac{dx}{dt} = 4t^3 \qquad \frac{dy}{dt} = 3t^5$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{(4t^3)^2 + (3t^5)^2}\,dt = \int_0^1 \sqrt{16t^6 + 9t^{10}}\,dt = \int_0^1 t^3\sqrt{16 + 9t^4}\,dt$$

Let $u = 16 + 9t^4$. Then $du = 36t^3 dt$ and $\frac{1}{36}du = t^3 dt$. So

$$L = \frac{1}{36}\int_{16}^{25} \sqrt{u}\,du = \frac{1}{36}\left[\frac{2u^{3/2}}{3}\right]_{16}^{25} = \frac{1}{54}(25^{3/2} - 16^{3/2}) = \frac{1}{54}(125 - 64) = \frac{61}{54}$$

8. The curve $y = x^3$ for $0 \le x \le 2$ is rotated about the x-axis. Find the surface area.

a. $\frac{\pi}{27}2^{3/2}$

b. $\frac{\pi}{12}(2^{3/2} - 1)$

c. $\frac{\pi}{27}(145^{3/2} - 1)$ correct choice

d. $\frac{\pi}{12}(145^{3/2} - 1)$

e. $\frac{\pi}{12}145^{3/2}$

Solution: $\frac{dy}{dx} = 3x^2$. The radius is $r = y = x^3$. So the surface area is:

$$A = \int 2\pi r\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^2 2\pi x^3\sqrt{1 + (3x^2)^2}\,dx = \int_0^2 2\pi x^3\sqrt{1 + 9x^4}\,dx$$

Let $u = 1 + 9x^4$. Then $du = 36x^3 dx$ and $\frac{1}{36}du = x^3 dx$. So

$$A = \frac{1}{36}\int_1^{145} 2\pi\sqrt{u}\,du = \left[\frac{\pi}{18}\frac{2u^{3/2}}{3}\right]_1^{145} = \frac{\pi}{27}(145^{3/2} - 1)$$

9. Compute $\int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta$.

a. $\frac{2}{5}$

b. $\frac{2}{3}$

c. $\frac{2}{15}$

d. $\frac{4}{15}$ correct choice

e. $\frac{8}{15}$

Solution: Let $u = \cos\theta$. Then $du = -\sin\theta \, d\theta$ and $\sin^2\theta = 1 - \cos^2\theta = 1 - u^2$. So
$\int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta = -\int_1^{-1} (1 - u^2)u^2 \, du = -\left[\frac{u^3}{3} - \frac{u^5}{5} \right]_1^{-1} = -2\left(\frac{-1}{3} - \frac{-1}{5} \right) = \frac{4}{15}$

10. Compute $\int_{-\pi/4}^{\pi/4} \tan^4\theta \sec^2\theta \, d\theta$.

a. $\frac{2}{5}$ correct choice

b. $\frac{2}{3}$

c. $\frac{2}{15}$

d. $\frac{4}{15}$

e. $\frac{8}{15}$

Solution: Let $u = \tan\theta$. Then $du = \sec^2\theta \, d\theta$. So
$\int_{-\pi/4}^{\pi/4} \tan^4\theta \sec^2\theta \, d\theta = \int_{-1}^{1} u^4 \, du = \left[\frac{u^5}{5} \right]_{-1}^{1} = \left(\frac{1}{5} - \frac{-1}{5} \right) = \frac{2}{5}$

11. Compute $\int_0^{\pi/4} \tan^3\theta \sec^3\theta \, d\theta$.

a. $\frac{2}{15}\left(\sqrt{2} - 1\right)$

b. $\frac{2}{15}\left(\sqrt{2} + 1\right)$ correct choice

c. $\frac{1}{15}\left(\sqrt{2} + 1\right)$

d. $\frac{1}{15}\left(\sqrt{2} - 1\right)$

e. $\frac{2}{15}\left(1 - \sqrt{2}\right)$

Solution: Let $u = \sec\theta$. Then $du = \sec\theta \tan\theta \, d\theta$ and $\tan^2\theta = \sec^2\theta - 1 = u^2 - 1$. So
$\int_0^{\pi/4} \tan^3\theta \sec^3\theta \, d\theta = \int_1^{\sqrt{2}} (u^2 - 1)u^2 \, du = \left[\frac{u^5}{5} - \frac{u^3}{3} \right]_1^{\sqrt{2}} = \left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} \right) - \left(\frac{1}{5} - \frac{1}{3} \right)$
$= \frac{12 - 10}{15}\sqrt{2} - \frac{3 - 5}{15} = \frac{2}{15}\left(\sqrt{2} + 1\right)$

12. Compute $\int \frac{1}{(9+x^2)^{3/2}}\,dx$

a. $\frac{1}{9\sqrt{9+x^2}} + C$

b. $\frac{x}{9\sqrt{9+x^2}} + C$ correct choice

c. $\frac{1}{3\sqrt{9+x^2}} + C$

d. $\frac{\sqrt{9+x^2}}{9x} + C$

e. $\frac{\sqrt{9+x^2}}{3x} + C$

Solution: Let $x = 3\tan\theta$. Then $dx = 3\sec^2\theta\,d\theta$. So

$$I = \int \frac{1}{(9+x^2)^{3/2}}\,dx = \int \frac{1}{(9+9\tan^2\theta)^{3/2}}\,3\sec^2\theta\,d\theta = \frac{1}{9}\int \frac{1}{(1+\tan^2\theta)^{3/2}}\sec^2\theta\,d\theta = \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta$$
$$= \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C$$

Draw a triangle with opposite side x, adjacent side 3 and hypotenous $\sqrt{9+x^2}$. So

$$I = \frac{x}{9\sqrt{9+x^2}} + C$$

13. Compute $\int \frac{1}{\sqrt{x^2+4x-5}}\,dx$.

a. $\frac{1}{x}\ln\left|x + \sqrt{x^2+4x-5}\right| + C$

b. $\ln\left|x - \sqrt{x^2+4x-5}\right| + C$

c. $\ln\left|x + 2 + \sqrt{x^2+4x-5}\right| + C$ correct choice

d. $\ln\left|x + 2 - \sqrt{x^2+4x-5}\right| + C$

e. $\ln\left|\frac{3}{x+2} + \frac{\sqrt{x^2+4x-5}}{3}\right| + C$

Solution: $x^2+4x-5 = (x+2)^2 - 9$. Let $x + 2 = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\,d\theta$. So

$$I = \int \frac{1}{\sqrt{(x+2)^2-9}}\,dx = \int \frac{1}{\sqrt{9\sec^2\theta - 9}}\,3\sec\theta\tan\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$

Draw a triangle with hypotenous $x+2$, adjacent side 3 and opposite side $\sqrt{(x+2)^2-9}$.

Then $\sec\theta = \frac{x+2}{3}$ and $\tan\theta = \frac{\sqrt{(x+2)^2-9}}{3}$. So

$$I = \ln\left|\frac{x+2+\sqrt{(x+2)^2-9}}{3}\right| + C = \ln\left|x+2+\sqrt{x^2+4x-5}\right| + C - \ln 3$$

Work Out: (Points indicated. Part credit possible. Show all work.)

14. $(20$ points$)$ A 10 cm bar has linear density $\delta = e^{-x}$ g/cm where x is measured from one end.

a. Find the total mass of the bar.

Solution: $M = \int \delta\,dx = \int_0^{10} e^{-x}\,dx = [-e^{-x}]_0^{10} = -e^{-10} + e^0 = 1 - e^{-10}$

b. Find the center of mass of the bar.

Solution: $M_1 = \int x\delta\,dx = \int_0^{10} xe^{-x}\,dx$ Use parts with $\begin{matrix} u = x & dv = e^{-x}\,dx \\ du = dx & v = -e^{-x} \end{matrix}$

$$M_1 = \left[-xe^{-x} + \int e^{-x}\,dx\right]_0^{10} = [-xe^{-x} - e^{-x}]_0^{10} = (-10e^{-10} - e^{-10}) - (-e^0) = 1 - 11e^{-10}$$

$$\bar{x} = \frac{M_1}{M} = \frac{1 - 11e^{-10}}{1 - e^{-10}}$$

15. $(10$ points$)$ Compute $\int x \arctan x\,dx$.

HINT: To complete the last integral, add and subtract 1 in the numerator.

Solution: Parts with $\begin{matrix} u = \arctan x & dv = x\,dx \\ du = \dfrac{1}{1+x^2}\,dx & v = \dfrac{1}{2}x^2 \end{matrix}$. Then

$$\int x \arctan x\,dx = \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx = \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\,dx$$
$$= \frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + C$$

16. $(15$ points$)$ Compute $\int e^{3x} \cos 4x\,dx$.

Solution: Use parts with $\begin{matrix} u = \cos 4x & dv = e^{3x}\,dx \\ du = -4\sin 4x\,dx & v = \dfrac{1}{3}e^{3x} \end{matrix}$. Then

$$I = \int e^{3x} \cos 4x\,dx = \frac{1}{3}e^{3x} \cos 4x + \frac{4}{3}\int e^{3x} \sin 4x\,dx$$

Next use parts with $\begin{matrix} u = \sin 4x & dv = e^{3x}\,dx \\ du = 4\cos 4x\,dx & v = \dfrac{1}{3}e^{3x} \end{matrix}$

$$I = \frac{1}{3}e^{3x} \cos 4x + \frac{4}{3}\left[\frac{1}{3}e^{3x} \sin 4x - \frac{4}{3}\int e^{3x} \cos 4x\,dx\right] = \frac{1}{3}e^{3x} \cos 4x + \frac{4}{9}e^{3x} \sin 4x - \frac{16}{9}I$$
$$I + \frac{16}{9}I = \frac{1}{3}e^{3x} \cos 4x + \frac{4}{9}e^{3x} \sin 4x$$
$$I = \frac{9}{25}\left(\frac{1}{3}e^{3x} \cos 4x + \frac{4}{9}e^{3x} \sin 4x\right) + C = \frac{3}{25}e^{3x} \cos 4x + \frac{4}{25}e^{3x} \sin 4x + C$$