Name________________________________

MATH 172H	Exam 2	Spring 2019
Sections 200	Solutions	P. Yasskin

11 Multiple Choice: (5 points each. No part credit.)

1. Consider the integrals:

$$A = \int_3^4 \frac{1}{(x-3)^{2/3}}dx \qquad B = \int_3^4 \frac{1}{(x-3)^{4/3}}dx \qquad C = \int_4^\infty \frac{1}{(x-3)^{2/3}}dx \qquad D = \int_4^\infty \frac{1}{(x-3)^{4/3}}dx$$

Which are finite? Which are infinite?

a. A and B are finite. C and D are infinite.
b. B and C are finite. A and D are infinite.
c. B and D are finite. A and C are infinite.
d. A and D are finite. B and C are infinite. correct choice
e. A and C are finite. B and D are infinite.

Solution: For large x, notice $\frac{1}{(x-3)^{4/3}}$ is more damped than $\frac{1}{x-3}$. So D is finite.
For large x, notice $\frac{1}{(x-3)^{2/3}}$ is less damped than $\frac{1}{x-3}$. So C is infinite.
Near $x = 3$, the behavior is reversed. So B is infinite and A is finite.

2. Compute $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx$.

a. π
b. $\frac{\pi}{2}$ correct choice
c. $\frac{\pi}{4}$
d. 0
e. divergent

Solution: $\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \left[\arcsin x\right]_0^1 = \arcsin 1 - \arcsin 0 = \frac{\pi}{2}$

1-11	/55	13	/22
12	/15	14	/12
		Total	/104

3. Which of the following terms does NOT belong in the general partial fraction expansion of

$$\frac{x^3 - 6x^2 + 7}{(x-4)(x-3)^2(x^2+4)(x^2+9)^3}$$

a. $\frac{A}{(x-4)}$

b. $\frac{B}{(x-3)^2}$

c. $\frac{Cx+D}{(x^2+9)}$

d. $\frac{Ex+F}{(x^2+9)^3}$

e. They all belong. correct choice

Solution: The general partial fraction expansion is

$$\frac{x^3 - 6x^2 + 7}{(x-4)(x-3)^3(x^2+4)(x^2+9)^4} = \frac{A}{(x-4)} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2} + \frac{Dx+E}{(x^2+4)}$$
$$+ \frac{Fx+G}{(x^2+9)} + \frac{Hx+I}{(x^2+9)^2} + + \frac{Jx+K}{(x^2+9)^3}$$

So they all belong.

4. In the partial fraction expansion $\frac{x}{(x-2)(x-3)^3} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{D}{(x-3)^3}$ which coefficient is INCORRECT?

a. $A = -2$

b. $B = 2$

c. $C = -3$ correct choice

d. $D = 3$

e. They are all correct.

Solution: Clear the denominator. Plug in $x = 2$ and $x = 3$.

$$x = A(x-3)^3 + B(x-2)(x-3)^2 + C(x-2)(x-3) + D(x-2)$$

$x = 2$: $2 = A(-1)^3 \Rightarrow A = -2$

$x = 3$: $3 = D(1) \Rightarrow D = 3$

Plug in $x = 0$ and $x = 1$ and use A and D.

$x = 0$: $0 = A(-3)^3 + B(-2)(-3)^2 + C(-2)(-3) + D(-2) = 54 - 18B + 6C - 6$
$\Rightarrow 3B - C = 8$

$x = 1$: $1 = A(-2)^3 + B(-1)(-2)^2 + C(-1)(-2) + D(-1) = 16 - 4B + 2C - 3$
$\Rightarrow 2B - C = 6$

Subtract the two equations to get $B = 2$. Substitute back to get $C = -2 \neq -3$

5. Find the location of the vertical tangents to the parametric curve:

$$x = t^3 - 3t \qquad y = t^2 - 4t$$

a. $(-2,-3)$ and $(2,5)$ only correct choice
b. $(-2,-3)$, $(2,-4)$ and $(2,5)$ only
c. $(-2,-3)$ and $(2,-4)$ only
d. $(2,-4)$ only
e. $(2,-4)$ and $(2,5)$ only

Solution: The vertical tangents occur when $\frac{dx}{dt} = 0$. Or $\frac{dx}{dt} = 3t^2 - 3 = 0$, or $t = \pm 1$.

At $t = 1$: $(x,y) = (t^3 - 3t, t^2 - 4t) = (1 - 3, 1 - 4) = (-2,-3)$

At $t = -1$: $(x,y) = (t^3 - 3t, t^2 - 4t) = (-1 + 3, 1 + 4) = (2,5)$

Note: $(2,-4)$ is a horizontal tangent.

6. The base of a solid is the region between $y = x^2$ and the x-axis for $0 \le x \le 3$. The cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

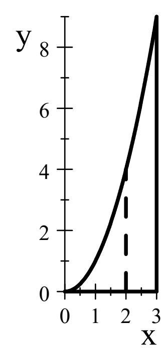
a. $\frac{3^4}{4}$
b. $\frac{3^5}{5}$ correct choice
c. 9
d. 27
e. 81

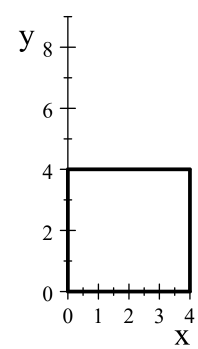
Solution: Here are plots of the base, a slice perpendicular to the x-axis and a cross section.

The area of the slice is $A = y^2 = (x^2)^2 = x^4$.

So the volume is

$$V = \int_0^3 A\,dx = \int_0^3 x^4\,dx = \left[\frac{x^5}{5}\right]_0^3 = \frac{3^5}{5}$$





7. The region between $y = x^2$ and the x-axis for $0 \le x \le 4$ is rotated about the y-axis. Find the volume swept out.

a. 8π

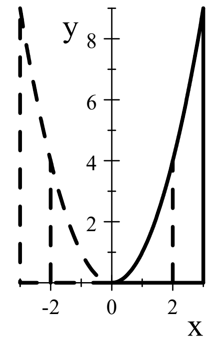
b. 16π

c. 32π

d. 64π

e. 128π correct choice

Solution: We do an x-integral. Here are plots of the region, a slice perpendicular to the x-axis and the shape rotated about the y-axis. The slice rotates into a cylinder. So the volume is



$$V = \int_0^4 2\pi r h\, dx = \int_0^4 2\pi(x)(x^2)\, dx$$

$$= 2\pi\left[\frac{x^4}{4}\right]_0^4 = 2 \cdot 4^3\pi = 128\pi$$

8. The region between $y = x^2$ and the x-axis for $0 \le x \le 4$ is rotated about the x-axis. Find the volume swept out.

a. $\dfrac{1024\pi}{5}$ correct choice

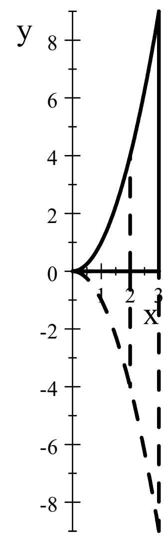
b. 64π

c. $\dfrac{64\pi}{3}$

d. 32π

e. $\dfrac{32\pi}{3}$

Solution: We do an x-integral. Here are plots of the region, a slice perpendicular to the x-axis and the shape rotated about the x-axis. The slice rotates into a disk. So the volume is



$$V = \int_0^4 \pi r^2\, dx = \int_0^4 \pi (x^2)^2\, dx = \pi\left[\frac{x^5}{5}\right]_0^4 = \frac{1024\pi}{5}$$

9. It takes a 40 N force to stretch a certain spring to 8 m from its rest position.
How much work does it take to stretch this spring from 1 m from rest to 9 m from rest.

a. 25 J
b. 50 J
c. 100 J
d. 200 J   correct choice
e. 400 J

Solution: $F = kx \qquad 40 = k8 \qquad \Rightarrow \qquad k = 5 \qquad \Rightarrow \qquad F = 5x$
$W = \int_1^9 F\,dx = \int_1^9 5x\,dx = \left[5\frac{x^2}{2}\right]_1^9 = \frac{5}{2}(81-1) = 200$ J

10. A 100 foot rope weighs $\delta = 2\,\frac{\text{lb}}{\text{foot}}$. It is hanging from the top of a 100 foot tall building.
How much work is done to pull it up to the top of the building.

a. 5000
b. 10000 correct choice
c. 20000
d. $\frac{100^3}{3}$
e. $2\frac{100^3}{3}$

Solution: Put the 0 of the y-axis at the top of the building and measure y downward. The piece of rope of length dy feet at a distance of y feet from the top is lifted a distance $D = y$ feet. Its weight is $dF = \delta\,dy = 2\,dy$. So the work done to lift the rope is

$$W = \int_0^{100} D\,dF = \int_0^{100} y\,2\,dy = \left[y^2\right]_0^{100} = 10000$$

11. Find the solution of the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$ satisfying $y(1) = 4$. Then $y(0) =$

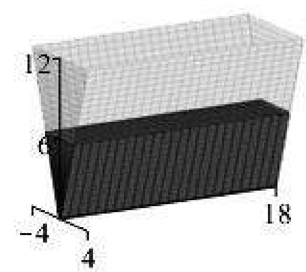
a. $\sqrt[3]{7}$
b. $\sqrt[3]{21}$
c. $\sqrt[3]{63}$ correct choice
d. $\sqrt[3]{65}$
e. $\sqrt[3]{195}$

Solution: We separate: $y^2\,dy = x^2\,dx \qquad \int y^2\,dy = \int x^2\,dx \qquad \frac{y^3}{3} = \frac{x^3}{3} + C$
We use the initial condition: $\frac{64}{3} = \frac{1}{3} + C \qquad C = 21 \qquad \frac{y^3}{3} = \frac{x^3}{3} + 21 \qquad y = \sqrt[3]{x^3 + 63}$
Then $y(0) = \sqrt[3]{63}$.

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (15 points) A water trough is 18 meters long. Its end is an isoceles triangle with vertex down whose width is 8 meters and height is 12 meters. The trough is filled with water to a depth of 6 meters. How much work is done to pump the water out the top of the tank? Answers can be given as a multiple of δg where δ is the densty of water g is the acceleration of gravity is g.

Solution: We put the 0 of the y-axis at the vertex of the triangle and measure y upward. The slice at height y is a rectangle with width w and length $l = 18$. Similar triangles say $\frac{w}{y} = \frac{8}{12} = \frac{2}{3}$ or $w = \frac{2}{3}y$. So the area of the slice is $A = lw = 18\frac{2}{3}y = 12y$. Its volume is $dV = A\,dy = 12y\,dy$. Its weight is $dF = \delta g dV = \delta g 12y\,dy$. This slab of water is lifted a distance $D = 12 - y$. So the work is



$$W = \int D\,dF = \int_0^6 (12-y)\delta g 12y\,dy = 12\delta g \int_0^6 (12y - y^2)\,dy = 12\delta g\left[6y^2 - \frac{y^3}{3}\right]_0^6$$
$$= 12\delta g\left(6^3 - \frac{6^3}{3}\right) = 12\delta g 6^3 \frac{2}{3} = 1728\delta g$$

13. $(22$ points$)$ A pot of syrup on a stove initially contains 4 cups of sugar in 16 gallons of water. Sugar water containing 2 cups of sugar per gallon is added at 3 gallons per hour. Pure water boils off at 1 gallon per hour. The syrup is kept well mixed and is drained at 2 gallons per hour. Let $S(t)$ be the cups of sugar in the pot after t hours.

a. Find the differential equation and initial condition satisfied by $S(t)$.

Solution: $S(0) = 4$

$$\frac{dS}{dt} = \underbrace{\frac{2 \text{ cups}}{\text{gal}} \cdot \frac{3 \text{ gal}}{\text{hr}}}_{\text{in}} - \underbrace{\frac{S(t) \text{ cups}}{16 \text{gal}} \cdot \frac{2 \text{ gal}}{\text{hr}}}_{\text{out}} \qquad \frac{dS}{dt} = 6 - \frac{1}{8}S$$

b. Solve for $S(t)$.

Solution: Method 1 Linear: Put the equation in standard form: $\frac{dS}{dt} + \frac{1}{8}S = 6$

Identify $P = \frac{1}{8}$ Find the integrating factor: $e^{\int P\,dt} = e^{t/8}$

Multiply thru by the integrating factor: $e^{t/8}\frac{dS}{dt} + \frac{1}{8}e^{t/8}S = 6e^{t/8}$ $\frac{d}{dt}\left(e^{t/8}S\right) = 6e^{t/8}$

Integrate and solve: $e^{t/8}S = 48e^{t/8} + C$ $S = 48 + Ce^{-t/8}$

Find the constant of integration: $4 = 48 + C$ $C = -44$

Substitute back: $S = 48 - 44e^{-t/8}$

Solution: Method 2 Separable: Separate: $\int \frac{dS}{6 - \frac{1}{8}S} = \int dt$

Integrate and solve: $-8\ln\left|6 - \frac{1}{8}S\right| = t + C$ $\ln\left|6 - \frac{1}{8}S\right| = -\frac{t}{8} - \frac{C}{8}$

$\left|6 - \frac{1}{8}S\right| = e^{-C/8}e^{-t/8}$ $6 - \frac{1}{8}S = \pm e^{-C/8}e^{-t/8} = Ae^{-t/8}$ $S = 48 - 8Ae^{-t/8}$

Find the constant of integration: $4 = 48 - 8A$ $A = \frac{44}{8}$

Substitute back: $S = 48 - 44e^{-t/8}$

c. After a very large time, how many cups of sugar will be in the pot?

Solution: After a very large time, $e^{-t/8} \to 0$. So $S = 48$.

14. (12 points) Given the partial fraction expansion $\frac{10x^2-60}{(x-4)^2(x^2+4)} = \frac{2}{x-4} + \frac{5}{(x-4)^2} + \frac{-2x-3}{x^2+4}$

Compute $\int \frac{10x^2-60}{(x-4)^2(x^2+4)}\,dx$.

Solution:

$$\int \frac{2}{x-4}\,dx = 2\ln|x-4| + C_1$$

$$\int \frac{5}{(x-4)^2}\,dx = \frac{-5}{x-4} + C_2$$

$$\int \frac{-2x}{x^2+4}\,dx = -\ln|x^2+4| + C_3$$

In the last integral, let $x = 2\tan\theta \qquad dx = 2\sec^2\theta\,d\theta$.

$$\int \frac{-3}{x^2+4}\,dx = \int \frac{-3}{4\tan^2\theta+4}2\sec^2\theta\,d\theta = \frac{-3}{2}\int \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta$$
$$= \frac{-3}{2}\int 1\,d\theta = \frac{-3}{2}\theta = \frac{-3}{2}\arctan\frac{x}{2} + C_4$$

So

$$\int \frac{10x^2-60}{(x-4)^2(x^2+4)}\,dx = 2\ln|x-4| - \frac{5}{x-4} - \ln|x^2+4| - \frac{3}{2}\arctan\frac{x}{2} + C$$