Name________________________________

MATH 172H Final Exam Spring 2019

Sections 200 Solutions P. Yasskin

15 Multiple Choice: (4 points each. No part credit.)

1. Compute $\int_1^e \frac{\ln x}{x^2}\,dx$.

a. 1
b. $1 - \frac{2}{e}$ correct choice
c. $-1 - \frac{2}{e}$
d. $\frac{2}{e} - 1$
e. $1 + \frac{2}{e}$

Solution: Integration by Parts: $u = \ln x \quad dv = \frac{1}{x^2}\,dx \quad du = \frac{1}{x}\,dx \quad v = \frac{-1}{x}$

$$\int_1^e \frac{\ln x}{x^2}\,dx = \frac{-1}{x}\ln x - \int \frac{-1}{x^2}\,dx = \left[\frac{-1}{x}\ln x - \frac{1}{x}\right]_1^e = \left(\frac{-1}{e} - \frac{1}{e}\right) - \left(-\frac{1}{1}\right) = 1 - \frac{2}{e}$$

2. Compute $\int \frac{\sqrt{x^2-4}}{x}\,dx$.

a. $\sqrt{x^2-4} - 2\,\mathrm{arcsec}\,\frac{x}{2} + C$ correct choice
b. $2\,\mathrm{arcsec}\,\frac{x}{2} - \sqrt{x^2-4} + C$
c. $\frac{\sqrt{x^2-4}}{2} - \ln\left(\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right) + C$
d. $\ln\left(\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right) - \frac{\sqrt{x^2-4}}{2} + C$
e. $\ln\left(x + \sqrt{x^2-4}\right) - \sqrt{x^2-4} + C$

Solution: $x = 2\sec\theta \quad dx = 2\sec\theta\tan\theta\,d\theta$

$$\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{\sqrt{4\sec^2\theta - 4}}{2\sec\theta}\,2\sec\theta\tan\theta\,d\theta = 2\int \tan^2\theta\,d\theta = 2\int(\sec^2\theta - 1)\,d\theta = 2\tan\theta - 2\theta + C$$

Draw a triangle with x on the hypotenuse, 2 on the adjacent side and $\sqrt{x^2-4}$ on the opposite side.

Then $\tan\theta = \frac{\sqrt{x^2-4}}{2}$ and $\int \frac{\sqrt{x^2-4}}{x}\,dx = \sqrt{x^2-4} - 2\,\mathrm{arcsec}\,\frac{x}{2} + C$

1-15	/60	18	/15
16	/10	19	/10
17	/10	Total	/105

3. The integral $\int_1^\infty \frac{1}{x^3 + \sqrt[3]{x}}\,dx$

a. converges by comparison to $\int_1^\infty \frac{1}{x^3}\,dx$. correct choice

b. diverges by comparison to $\int_1^\infty \frac{1}{x^3}\,dx$.

c. converges by comparison to $\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx$.

d. diverges by comparison to $\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx$.

Solution: x is larger than $\sqrt[3]{x}$ for large x. So we compare to $\int_1^\infty \frac{1}{x^3}\,dx$ which converges because:

$$\int_1^\infty \frac{1}{x^3}\,dx = \left[-\frac{1}{2x^2}\right]_1^\infty = 0 - -\frac{1}{2} = \frac{1}{2} \quad \text{which is finite.}$$

Since $\frac{1}{x^3 + \sqrt[3]{x}} < \frac{1}{x^3}$, (There's more in the bottom.) the integral $\int_1^\infty \frac{1}{x^3 + \sqrt[3]{x}}\,dx$ also converges.

4. Find the average value of the function $f(x) = \frac{1}{4+x^2}$ on the interval $[0,2]$.

a. 2π

b. $\frac{\pi}{2}$

c. $\frac{\pi}{4}$

d. $\frac{\pi}{8}$

e. $\frac{\pi}{16}$ correct choice

Solution: $\int_0^2 \frac{1}{4+x^2}\,dx = \left[\frac{1}{2}\arctan\frac{x}{2}\right]_0^2$ (If necessary, substitute $x = 2u$ or $x = 2\tan\theta$.)

Then $\int_0^2 \frac{1}{4+x^2}\,dx = \frac{1}{2}\arctan 1 - \frac{1}{2}\arctan 0 = \frac{\pi}{8}$ since $\arctan 1 = \frac{\pi}{4}$ and $\arctan 0 = 0$.

So $f_{\text{ave}} = \frac{1}{2}\int_0^2 \frac{1}{4+x^2}\,dx = \frac{1}{2}\frac{\pi}{8} = \frac{\pi}{16} \approx 0.2$

5. Find the center of mass of a bar which is 4 cm long and has density $\delta = 3x + 2x^2$ where x is measured from one end.

a. $\frac{17}{24}$

b. $\frac{24}{17}$

c. $\frac{25}{72}$

d. $\frac{72}{25}$ correct choice

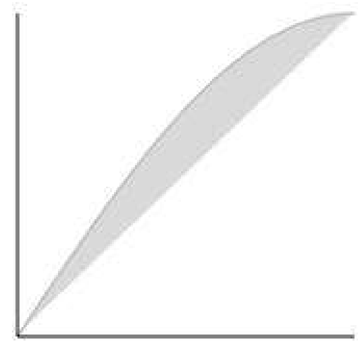
e. $\frac{200}{3}$

Solution: $M = \int_0^4 \delta\,dx = \int_0^4 (3x + 2x^2)\,dx = \left[\frac{3x^2}{2} + \frac{2x^3}{3}\right]_0^4 = 24 + \frac{128}{3} = \frac{200}{3}$

$M_1 = \int_0^4 x\delta\,dx = \int_0^4 (3x^2 + 2x^3)\,dx = \left[x^3 + \frac{x^4}{2}\right]_0^4 = 64 + 128 = 192$

$\bar{x} = \frac{M_1}{M} = 192 \cdot \frac{3}{200} = \frac{72}{25}$

6. The region between $y = \sin x$ and $y = \frac{2x}{\pi}$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the y-axis. Which integral gives the volume swept out?



a. $V = \int_0^{\pi/2} 2\pi x\left(\frac{2x}{\pi} - \sin x\right) dx$

b. $V = \int_0^{\pi/2} 2\pi\left(\sin^2 x - \frac{4x^2}{\pi^2}\right) dx$

c. $V = \int_0^{\pi/2} 2\pi x\left(\sin x - \frac{2x}{\pi}\right) dx$ correct choice

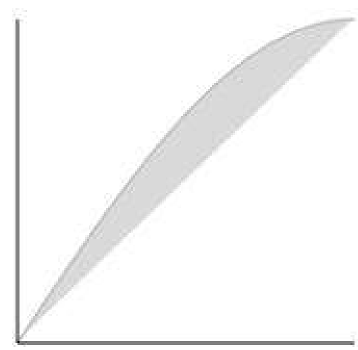
d. $V = \int_0^{\pi/2} \pi\left(\frac{4x^2}{\pi^2} - \sin^2 x\right) dx$

e. $V = \int_0^{\pi/2} \pi\left(\sin^2 x - \frac{4x^2}{\pi^2}\right) dx$

Solution: Do a x-integral. Slices are vertical. They rotate about the y-axis into cylinders.

$$V = \int_0^{\pi/2} 2\pi r h\, dx = \int_0^{\pi/2} 2\pi x\left(\sin x - \frac{2x}{\pi}\right) dx$$

7. The region between $y = \sin x$ and $y = \frac{2x}{\pi}$ for $0 \le x \le \frac{\pi}{2}$ is rotated about the x-axis. Which integral gives the volume swept out?



a. $V = \int_0^{\pi/2} 2\pi x\left(\frac{2x}{\pi} - \sin x\right) dx$

b. $V = \int_0^{\pi/2} 2\pi\left(\sin^2 x - \frac{4x^2}{\pi^2}\right) dx$

c. $V = \int_0^{\pi/2} 2\pi x\left(\sin x - \frac{2x}{\pi}\right) dx$

d. $V = \int_0^{\pi/2} \pi\left(\frac{4x^2}{\pi^2} - \sin^2 x\right) dx$

e. $V = \int_0^{\pi/2} \pi\left(\sin^2 x - \frac{4x^2}{\pi^2}\right) dx$ correct choice

Solution: Do an x-integral. Slices are vertical. They rotate about the x-axis into washers.

$$V = \int_0^{\pi/2} \pi(R^2 - r^2)\, dx = \int_0^{\pi/2} \pi\left(\sin^2 x - \frac{4x^2}{\pi^2}\right) dx$$

8. Solve the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$ with the initial condition $y(1) = 2$. Then $y(3) =$

a. -6
b. 4
c. $\sqrt[3]{7}$
d. $\sqrt[3]{20}$
e. $\sqrt[3]{34}$ correct choice

Solution: Separate: $\int y^2\, dy = \int x^2\, dx$ Integrate: $\frac{y^3}{3} = \frac{x^3}{3} + C$ Find C: $\frac{8}{3} = \frac{1}{3} + C$ $C = \frac{7}{3}$

Plug in and solve for y: $\frac{y^3}{3} = \frac{x^3}{3} + \frac{7}{3}$ $y = \sqrt[3]{x^3 + 7}$ Then $y(3) = \sqrt[3]{34}$.

9. For the differential equation, $\frac{1}{y}\frac{dy}{dx} = \frac{5}{xy} + \frac{3}{x}$, the integrating factor is

a. $-3\ln x$
b. $\frac{1}{x^3}$ correct choice
c. $-\frac{3}{x}$
d. x^3
e. There is no integrating factor since the equation is not linear.

Solution: In standard form the differential equation is $\frac{dy}{dx} - \frac{3}{x}y = \frac{5}{x}$.
We identify $P = -\frac{3}{x}$ So $\int P\,dx = -3\ln x = \ln x^{-3}$ and $I = e^{\int P\,dx} = e^{\ln x^{-3}} = x^{-3} = \frac{1}{x^3}$

10. Compute $L = \lim\limits_{n\to\infty}\left(\frac{n-3}{n-1}\right)^n$.

a. -2
b. e^{-2} correct choice
c. 1
d. e^2
e. ∞

Solution: $\ln L = \lim\limits_{n\to\infty} \ln\left(\frac{n-3}{n-1}\right)^n = \lim\limits_{n\to\infty} n\ln\left(\frac{n-3}{n-1}\right) = \lim\limits_{n\to\infty} \frac{\ln(n-3)-\ln(n-1)}{1/n}$
$\overset{l'H}{=} \lim\limits_{n\to\infty} \frac{\frac{1}{n-3} - \frac{1}{n-1}}{-1/n^2} = \lim\limits_{n\to\infty} \frac{2}{(n-3)(n-1)}(-n^2) = -2$ $\qquad L = e^{\ln L} = e^{-2}$

11. A sequence a_n is defined recursively by $a_{n+1} = \frac{(a_n)^2+2}{3}$ and $a_1 = 4$. Find $\lim\limits_{n\to\infty} a_n$.

a. 1
b. 2
c. 4
d. 16
e. ∞ correct choice

Solution: If the sequence has a limit $L = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$, then $L = \frac{L^2+2}{3}$ or $0 = L^2 - 3L + 2 = (L-1)(L-2)$. So the limit (if it exists) must be 1 or 2. The first few terms are $a_1 = 4$, $a_2 = \frac{4^2+2}{3} = 6$, $a_3 = \frac{36+2}{3} = \frac{38}{3} \approx 12.7$.
So the sequence is increasing from 4 and cannot have a limit of 1 or 2. So $\lim\limits_{n\to\infty} a_n = \infty$.

12. The series $\sum_{n=2}^{\infty} \frac{3n}{n^3 - 2}$

a. converges by Simple Comparison with $\sum_{n=2}^{\infty} \frac{3}{n^2}$.

b. diverges by Simple Comparison with $\sum_{n=2}^{\infty} \frac{3}{n^2}$.

c. converges by Limit but not Simple Comparison with $\sum_{n=2}^{\infty} \frac{3}{n^2}$. correct choice

d. diverges by Limit but not Simple Comparison with $\sum_{n=2}^{\infty} \frac{3}{n^2}$.

e. converges by the Ratio Test.

Solution: The series $\sum_{n=1}^{\infty} \frac{3}{n^2}$ converges because it is a p-series with $p = 2 > 1$. But $\frac{3n}{n^3 - 2} > \frac{3}{n^2}$, so Simple Comparison fails. However, $\lim_{n\to\infty} \frac{a_n}{b_b} = \lim_{n\to\infty} \frac{3n}{n^3 - 2} \frac{n^2}{3} = 1$ which is > 0 and $< \infty$. So the Limit Comparison Test says $\sum_{n=2}^{\infty} \frac{3n}{n^3 - 2}$ also converges. The Ratio Test fails.

13. Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}(x-3)^n$.

a. 0
b. 1
c. 2
d. 4 correct choice
e. ∞

Solution: Ratio Test:
$$\rho = \lim_{n\to\infty} \frac{((n+1)!)^2|x-3|^{n+1}}{(2n+2)!} \frac{(2n)!}{(n!)^2|x-3|^n} = |x-3| \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{|x-3|}{4} < 1. \quad \text{So} \quad R = 4.$$

14. The series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n(n^3 + \sqrt[3]{n})}$ has radius of convergence $R = 3$. Find its interval of convergence.

a. $(-1,5)$
b. $[-1,5)$
c. $(-1,5]$
d. $[-1,5]$ correct choice

Solution:

At $x = -1$: $\sum_{n=0}^{\infty} \frac{(-3)^n}{3^n(n^3 + \sqrt[3]{n})} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^3 + \sqrt[3]{n}}$ which **converges** by the Alternating Series Test.

At $x = 5$: $\sum_{n=0}^{\infty} \frac{(3)^n}{3^n(n^3 + \sqrt[3]{n})} = \sum_{n=0}^{\infty} \frac{1}{n^3 + \sqrt[3]{n}}$ which **converges** by a simple comparison with $\sum_{n=0}^{\infty} \frac{1}{n^3}$ which is a convergent p-series since $p = 3 > 1$.

15. For the function $f(x) = \cos(x^3)$ which of the following is FALSE?.

a. $f^{(27)}(0) = 0$
b. $f^{(28)}(0) = 0$
c. $f^{(29)}(0) = 0$
d. $f^{(30)}(0) = 0$ correct choice
e. $f^{(31)}(0) = 0$

Solution: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $\quad \cos x^3 = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$

In $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$, $f^{(k)}(0)$ is in the coefficient of x^k, but $k = 6n$.

So $f^{(k)}(0) = 0$ except for $k = 30$.

Work Out: (Points indicated. Part credit possible. Show all work.)

16. (10 points) Compute $\int \frac{2}{x^3 - x}\, dx$.

a. Find the general partial fraction expansion. (Do not find the coefficients.)

Solution: $\frac{2}{x^3 - x} = \frac{2}{x(x-1)(x+1)} = \underline{\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}}$

b. Find the coefficients and plug them back into the expansion.

Solution: Clear the denominator: $2 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

Plug in $x = 0$: $\quad 2 = A(-1) \quad A = -2$

Plug in $x = 1$: $\quad 2 = B(2) \quad B = 1$

Plug in $x = -1$: $\quad 2 = C(2) \quad C = 1$

$$\frac{2}{x^3 - x} = \underline{-\frac{2}{x} + \frac{1}{x-1} + \frac{1}{x+1}}$$

c. Compute the integral.

Solution: $\int \frac{2}{x^3 - x}\, dx = \int -\frac{2}{x}\, dx + \int \frac{1}{x-1}\, dx + \int \frac{1}{x+1}\, dx = \underline{-2\ln|x| + \ln|x-1| + \ln|x+1| + C}$

17. (10 points) The curve $(x,y) = \left(\frac{1}{2}t^2, \frac{1}{3}t^3\right)$ for $0 \le t \le \sqrt{3}$ is rotated about the y-axis. Find the area of the surface swept out.

Solution: The arc length differential is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(t)^2 + (t^2)^2}\, dt = t\sqrt{1+t^2}\, dt$$

The radius of revolution is $r = x = \frac{1}{2}t^2$. So the surface area is

$$A = \int 2\pi r\, ds = \int_0^{\sqrt{3}} 2\pi \frac{1}{2}t^2\, t\sqrt{1+t^2}\, dt = \int_0^{\sqrt{3}} \pi t^3 \sqrt{1+t^2}\, dt$$

Let $u = 1 + t^2$. Then $du = 2t\,dt$ and $t^2 = u - 1$. So

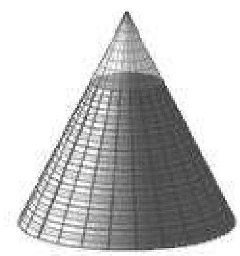
$$A = \frac{\pi}{2}\int_1^4 (u-1)\sqrt{u}\, du = \frac{\pi}{2}\left[\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right]_1^4 = \pi\left[\frac{u^{5/2}}{5} - \frac{u^{3/2}}{3}\right]_1^4$$

$$= \pi\left[\frac{32}{5} - \frac{8}{3}\right] - \pi\left[\frac{1}{5} - \frac{1}{3}\right] = 8\pi\left[\frac{4}{5} - \frac{1}{3}\right] - \pi\left[\frac{1}{5} - \frac{1}{3}\right] = 8\pi\frac{12-5}{15} - \pi\frac{-2}{15} = \frac{58}{15}\pi$$

18. (15 points) A water tank has the shape of a cone with the vertex at the top. Its height is $H = 30$ ft and its radius is $R = 10$ ft.

It is filled with salt water to a depth of 20 ft which weighs $\delta = 63\ \frac{\text{lb}}{\text{ft}^3}$.

Find the work done to pump the water out the top of the tank.



Solution: Put the y-axis measuring down from the top. The slice which is a distance y down from the top is a circle of radius r. By similar triangles, $\frac{r}{y} = \frac{R}{H} = \frac{10}{30} = \frac{1}{3}$. So $r = \frac{1}{3}y$. The area is $A = \pi r^2 = \frac{\pi y^2}{9}$ and the volume of the slice of thickness dy is $dV = A\,dy = \frac{\pi y^2}{9}dy$. It weighs $dF = \delta\, dV = 63\frac{\pi y^2}{9}dy = 7\pi y^2\, dy$. It is lifted a distance $D = y$. There is water between $y = 10$ and $y = 30$. So the work done is

$$W = \int_{10}^{30} D\,dF = \int_{10}^{30} y\,7\pi y^2\, dy = \left[7\pi\frac{y^4}{4}\right]_{10}^{30} = \frac{7}{4}\pi(30^4 - 10^4) = 1\,400\,000\pi \text{ ft-lb}$$

19. (10 points) The Maclaurin series for $f(x) = \frac{1}{1-x}$ is the geometric series $\sum_{n=0}^{\infty} x^n$ which converges for $|x| < 1$. For $x < 0$, the series is alternating; for $x > 0$ it is positive. We will approximate the series on the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$ by the 9^{th} degree Maclaurin polynomial which is the 9^{th} partial sum $S_9(x) = \sum_{n=0}^{9} x^n$. The error in this approximation is the remainder $R_9(x) = f(x) - S_9(x)$, which of course depends on the value of x.

a. Find the Alternating Series bound on the remainder for $x \in \left(-\frac{1}{2}, 0\right)$.
NOTE: This should be a single number which works for all values of x in the interval.

Solution: Since the series is alternating on this interval, the error is bounded by the next term:

$$|R_n(x)| < |x^{10}|$$

For $x \in \left(-\frac{1}{2}, 0\right)$ the largest this could be is

$$|R_n(x)| < \left|\frac{1}{2^{10}}\right| \approx 10^{-3}$$

b. The Taylor Remainder Inequality says

$$|R_n(x)| < \frac{M}{(n+1)!}|x|^{n+1} \qquad \text{where} \quad \dot{M} \geq f^{(n+1)}(c) \quad \text{for all} \quad c \quad \text{between} \quad 0 \quad \text{and} \quad x.$$

Find the Taylor Remainder bound on the remainder for $x \in \left(0, \frac{1}{2}\right)$.
NOTE: This should be a single number which works for all values of x in the interval.

Solution: $f(x) = \frac{1}{1-x}$ $\quad f'(x) = \frac{1}{(1-x)^2}$ $\quad f''(x) = \frac{2}{(1-x)^3}$ $\quad f^{(10)}(x) = \frac{10!}{(1-x)^{11}}$

The largest value on $\left(0, \frac{1}{2}\right)$ occurs at $x = \frac{1}{2}$. So we take

$M = f^{(10)}\left(\frac{1}{2}\right) = \frac{10!}{\left(1-\frac{1}{2}\right)^{11}} = 2^{11}10!$.

Also the largest value of $|x|$ is $\frac{1}{2}$. So

$$|R_9(x)| < \frac{M}{10!}|x|^{10} \leq \frac{2^{11}10!}{10!}\left(\frac{1}{2}\right)^{10} = 2$$