Name____

MATH 172H	Final Exam	Fall 2019
Sections 202	Solutions	P. Yasskin

1-13	/65	16	/20
15	/10	17	/10
		Total	/105

Multiple Choice: (5 points each. No part credit.)

1. Compute $\int_0^{\pi/4} \cos\theta \sin^3\theta \, d\theta$

a. $\frac{1}{2}$

b. $\frac{1}{4}$

c. $\frac{1}{8}$

d. $\frac{1}{16}$ correct choice

e. $\frac{1}{32}$

Solution: $u = \sin\theta \qquad du = \cos\theta \, d\theta \qquad \int_0^{\pi/4} \cos\theta \sin^3\theta \, dx = \left[\frac{\sin^4\theta}{4} \right]_0^{\pi/4} = \frac{1}{4} \frac{1}{\sqrt{2}^4} = \frac{1}{16}$

2. Compute $\int_0^{\ln 2} x e^{-x} \, dx$

a. $\frac{1}{2} \ln 2 + \frac{1}{2}$

b. $-\frac{1}{2} \ln 2 + \frac{1}{2}$ correct choice

c. $\frac{1}{2} \ln 2 - \frac{1}{2}$

d. $-\frac{1}{2} \ln 2 - \frac{1}{2}$

e. Divergent

Solution: Parts: $u = x \qquad dv = e^{-x} dx$

$du = dx \qquad v = -e^{-x}$

$$\int_0^{\ln 2} x e^{-x} \, dx = \left[-x e^{-x} + \int e^{-x} \, dx \right]_0^{\ln 2} = \left[-x e^{-x} - e^{-x} \right]_0^{\ln 2} = (-\ln 2 e^{-\ln 2} - e^{-\ln 2}) - (-1) = -\frac{1}{2} \ln 2 + \frac{1}{2}$$

3. Find the average value of $f(x) = \cos x$ on the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$.

a. $\frac{2\sqrt{2}}{\pi}$ correct choice

b. $\frac{\sqrt{2}}{\pi}$

c. $\sqrt{2}$

d. $\frac{1}{\sqrt{2}}$

e. $\frac{\pi}{\sqrt{2}}$

Solution: $f_{\text{ave}} = \frac{1}{\pi/2} \int_{-\pi/4}^{\pi/4} \cos x \, dx = \frac{2}{\pi} \left[\sin x \right]_{-\pi/4}^{\pi/4} = \frac{2}{\pi} \left(\frac{1}{\sqrt{2}} \right) - \frac{2}{\pi} \left(-\frac{1}{\sqrt{2}} \right) = \frac{4}{\pi\sqrt{2}} = \frac{2\sqrt{2}}{\pi}$

4. The partial fraction decomposition of $\dfrac{1}{x^2 - x}$ is

a. $\dfrac{1}{x-1} + \dfrac{1}{x}$

b. $\dfrac{1}{x-1} - \dfrac{1}{x}$ correct choice

c. $\dfrac{1}{x} - \dfrac{1}{x-1}$

d. $\dfrac{1}{x} + \dfrac{1}{x+1}$

e. $\dfrac{1}{x+1} - \dfrac{1}{x}$

Solution: $\dfrac{1}{x^2 - x} = \dfrac{1}{x(x-1)} = \dfrac{A}{x} + \dfrac{B}{x-1} \quad \Rightarrow \quad 1 = A(x-1) + Bx = (A+B)x - A$

$\Rightarrow \quad A + B = 0, \qquad -A = 1 \quad \Rightarrow \quad A = -1 \qquad B = 1 \quad \Rightarrow \quad \dfrac{1}{x^2 - x} = \dfrac{-1}{x} + \dfrac{1}{x-1}$

5. Compute $\displaystyle\int_3^{3\sqrt{2}} \frac{\sqrt{x^2 - 9}}{x}\, dx$.

a. $\dfrac{1}{\sqrt{2}} - 1$

b. $1 - \dfrac{1}{\sqrt{2}}$

c. $3\left(1 - \dfrac{\pi}{4}\right)$ correct choice

d. $3\left(\dfrac{\pi}{4} - 1\right)$

e. ∞

Solution: $x = 3\sec\theta \qquad dx = 3\sec\theta\tan\theta\, d\theta$

$$\int_3^{3\sqrt{2}} \frac{\sqrt{x^2 - 9}}{x}\, dx = \int_0^{\pi/4} \frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta} 3\sec\theta\tan\theta\, d\theta = 3\int_0^{\pi/4} \tan^2\theta\, d\theta$$

$$= 3\int_0^{\pi/4} \sec^2\theta - 1\, d\theta = 3\Big[\tan\theta - \theta\Big]_0^{\pi/4} = 3\left(1 - \frac{\pi}{4}\right)$$

6. Find the arc length of the curve $y = \dfrac{x^2}{4} - \dfrac{\ln x}{2}$ between $x = 1$ and $x = e$.

a. $\dfrac{e^2}{4} - \dfrac{1}{2}$

b. $\dfrac{e^2}{2} - \dfrac{1}{2}$

c. $\dfrac{e^2}{2} + \dfrac{1}{2}$

d. $\dfrac{e^2}{4} - \dfrac{1}{4}$

e. $\dfrac{e^2}{4} + \dfrac{1}{4}$ correct choice

Solution: $1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \left(\dfrac{x}{2} - \dfrac{1}{2x}\right)^2 = 1 + \dfrac{1}{4}x^2 - \dfrac{1}{2} + \dfrac{1}{4x^2} = \dfrac{1}{4}x^2 + \dfrac{1}{2} + \dfrac{1}{4x^2} = \left(\dfrac{x}{2} + \dfrac{1}{2x}\right)^2$

$$L = \int_1^e \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^e \left(\frac{x}{2} + \frac{1}{2x}\right) dx = \left[\frac{x^2}{4} + \frac{\ln x}{2}\right]_1^e = \left(\frac{e^2}{4} + \frac{1}{2}\right) - \left(\frac{1}{4}\right) = \frac{e^2}{4} + \frac{1}{4}$$

7. The base of a solid is the semi-circle between $y = \sqrt{9-x^2}$ and the x-axis. The cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

a. $\dfrac{9}{2}\pi$

b. 9π

c. 9

d. 18

e. 36 correct choice

Solution: This is a x-integral. The side of the square is $s = \sqrt{9-x^2}$. So the area is $A = s^2 = 9 - x^2$. So the volume is

$$V = \int_{-3}^{3} A\,dx = \int_{-3}^{3} (9-x^2)\,dx = \left[9x - \frac{x^3}{3}\right]_{-3}^{3} = 2(27-9) = 36$$

8. The region bounded by $x = 0$, $x = \cos y$, $y = 0$, $y = \frac{\pi}{4}$ is rotated about the x-axis. Which integral gives the volume of the solid of revolution?

a. $\int_0^{\pi/4} 2\pi \cos^2 y\,dy$

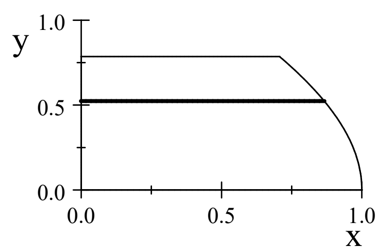
b. $\int_0^{\sqrt{2}/2} 2\pi x \arccos x\,dx$

c. $\int_0^{\pi/4} 2\pi y \cos y\,dy$ correct choice

d. $\int_0^{\sqrt{2}/2} \pi(\cos^2 x - x^2)\,dx$

e. $\int_0^{\pi/4} 2\pi y^2\,dy$

Solution:



y-integral cylinders

$r = y \qquad h = x = \cos y$

$$V = \int_0^{\pi/4} 2\pi r h\,dy = \int_0^{\pi/4} 2\pi y \cos y\,dy$$

9. As n approaches infinity, the sequence $a_n = \dfrac{1-\cos n}{n^2}$

a. converges to $-\dfrac{1}{2}$

b. converges to 0 correct choice

c. converges to $\dfrac{1}{2}$

d. converges to 1

e. diverges

Solution: Since $0 \le \dfrac{1-\cos n}{n^2} \le \dfrac{2}{n^2}$ and $\lim\limits_{n\to\infty} \dfrac{2}{n^2} = 0$, the Squeeze Theorem says $\lim\limits_{n\to\infty} \dfrac{1-\cos n}{n^2} = 0$.

10. $\sum_{n=2}^{\infty} \frac{3^n}{4^{n-1}} =$

a. $\frac{9}{7}$

b. 3

c. 4

d. 9 correct choice

e. Diverges

Solution: $a = \frac{3^2}{4^{2-1}} = \frac{9}{4} \qquad r = \frac{3}{4} \qquad |r| < 1 \qquad \sum_{n=2}^{\infty} \frac{3^n}{4^{n-1}} = \frac{\frac{9}{4}}{1 - \frac{3}{4}} = \frac{9}{4-3} = 9$

11. Compute $\sum_{n=1}^{\infty}\left(\frac{n}{n+1} - \frac{n+1}{n+2}\right)$

a. $-\frac{1}{2}$ correct choice

b. $\frac{1}{2}$

c. 1

d. 2

e. Divergent

Solution: $S_k = \sum_{n=1}^{k}\left(\frac{n}{n+1} - \frac{n+1}{n+2}\right) = \left(\frac{1}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{3}{4}\right) + \cdots + \left(\frac{k}{k+1} - \frac{k+1}{k+2}\right) = \frac{1}{2} - \frac{k+1}{k+2}$

$S = \lim_{k\to\infty}\left(\frac{1}{2} - \frac{k+1}{k+2}\right) = -\frac{1}{2}$

12. Compute $\lim_{x\to 0} \frac{\sin(2x) - 2x}{3x^3}$

a. $-\frac{1}{9}$

b. -4

c. $-\frac{8}{9}$

d. $-\frac{4}{3}$

e. $-\frac{4}{9}$ correct choice

Solution: $\lim_{x\to 0} \frac{\sin(2x) - 2x}{3x^3} = \lim_{x\to 0} \frac{\left(2x - \frac{(2x)^3}{3!} + \cdots\right) - 2x}{3x^3} = \lim_{x\to 0}\left(-\frac{(2x)^3}{3!3x^3} + \cdots\right) = -\frac{8}{6 \cdot 3} = -\frac{4}{9}$

13. If $g(x) = \cos(x^2)$, what is $g^{(8)}(0)$, the 8^{th} derivative at zero?
HINT: What is the coefficient of x^8 in the Maclaurin series for $\cos(x^2)$?

a. $\dfrac{1}{8 \cdot 7 \cdot 6 \cdot 5}$
b. $4!$
c. $\dfrac{1}{4!}$
d. $8 \cdot 7 \cdot 6 \cdot 5$ correct choice
e. $\dfrac{1}{8!}$

Solution: On the one hand, the Maclaurin series for $\cos(t)$ is $\cos(t) = 1 - \dfrac{t^2}{2!} + \dfrac{t^4}{4!} - \cdots$.
So the Maclaurin series for $\cos(x^2)$ is $\cos(x^2) = 1 - \dfrac{x^4}{2!} + \dfrac{x^8}{4!} - \cdots$.
On the other hand the Maclaurin series for any function $g(x)$ is

$$g(x) = g(0) + g'(0)x + \cdots + \frac{g^{(8)}(0)}{8!}x^8 + \cdots$$

Since these must be equal, the coefficients of x^8 must be equal: $\dfrac{g^{(8)}(0)}{8!} = \dfrac{1}{4!}$
So $g^{(8)}(0) = \dfrac{8!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$

Work Out (Points indicated. Part credit possible. Show all your work.)

14. (10 points) Estimate $\int_0^{0.1} \sin(x^2)\,dx$ to within an error of $|E| < 10^{-6}$.
Be sure to say why the error is less than 10^{-6}.
HINT: Use a Maclaurin series.

Solution: $\sin x = x - \dfrac{x^3}{6} + \cdots$ $\quad \sin x^2 = x^2 - \dfrac{x^6}{6} + \cdots$ $\quad \int_0^x \sin(x^2)\,dx = \dfrac{x^3}{3} - \dfrac{x^7}{42} + \cdots$
Using 1 term: $\int_0^{0.1} \sin(x^2)\,dx \approx \dfrac{(0.1)^3}{3}$.
Since this is an alternating series, the error is less than the next term $|E| < \dfrac{(0.1)^7}{42} < 10^{-6}$

15. (20 points) Find the radius and interval of convergence of $\sum_{n=2}^{\infty} \frac{(x-3)^n}{4\ln n}$. Be sure to check the endpoints. Name or state any test you use and check the conditions.

Solution: Ratio Test: $a_n = \frac{(x-3)^n}{4\ln n}$ $a_{n+1} = \frac{(x-3)^{n+1}}{4\ln(n+1)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{4\ln(n+1)}\frac{4\ln n}{(x-3)^n}\right| = |x-3|\lim_{n\to\infty}\left|\frac{\ln n}{\ln(n+1)}\right| = |x-3|\lim_{n\to\infty}\left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right| = |x-3| < 1$$

Radius: $R = 1$.

Solution: Converges if $2 < x < 4$.

At $x = 2$ the series is $\sum_{n=2}^{\infty} \frac{(-1)^n}{4\ln n}$ which converges by the Alternating Series Test.

The $(-1)^n$ says it alternates. The $\frac{1}{4\ln n}$ decreases and $\lim_{n\to\infty} \frac{1}{4\ln n} = 0$.

At $x = 4$ the series is $\sum_{n=2}^{\infty} \frac{1}{4\ln n}$. We apply the Comparison Test with $\sum_{n=2}^{\infty} \frac{1}{3n}$ which is a divergent harmonic series. Since $n > \ln n$ we have $\frac{1}{3n} < \frac{1}{3\ln n}$ and hence $\sum_{n=2}^{\infty} \frac{1}{3\ln n}$ also diverges.

Interval: $[2,4)$.

16. (10 points) The Taylor Remainder Theorem states:

If a function $f(x)$ is approximated by its k^{th} degree Taylor polynomial

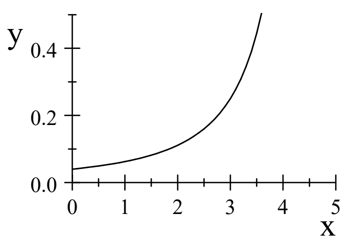
$T_k f(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$ then the Taylor remainder $R_k f(x) = f(x) - T_k f(x)$

is $R_k f(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}$ for some c between a and x.

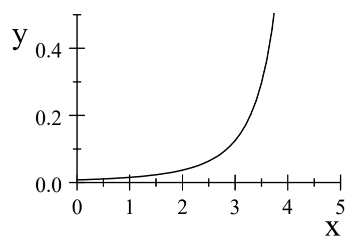
The Maclaurin series for $f(x) = \frac{1}{5-x}$ is

$$Tf(x) = \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}} = \frac{1}{5} + \frac{1}{25}x + \frac{1}{125}x^2 + \frac{1}{625}x^3 + O(x^4)$$

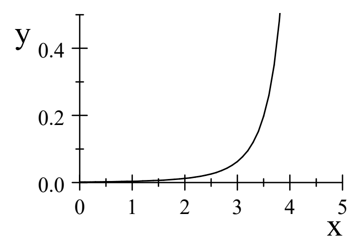
If the 2^{nd} degree Taylor polynomial, $T_2 f(x) = \frac{1}{5} + \frac{1}{25}x + \frac{1}{125}x^2$, is used to approximate $\frac{1}{5-x}$ at $x = 3$, use the Taylor Remainder Theorem to find a bound on the absolute value of the Taylor remainder, $|R_2 f(3)|$. Here are three plots:



$\frac{1}{(5-x)^2}$



$\frac{1}{(5-x)^3}$



$\frac{1}{(5-x)^4}$

Solution: $R_2 f(x) = \frac{f'''(c)}{3!}x^3$ for some c between 0 and x. Here $x = 3$. We compute the derivatives of $f(x) = \frac{1}{5-x}$:

$$f'(x) = \frac{1}{(5-x)^2} \qquad f''(x) = \frac{2}{(5-x)^3} \qquad f'''(x) = \frac{6}{(5-x)^4}$$

Since $\frac{6}{(5-x)^4}$ is increasing and $c \in [0,3]$, $f'''(c) = \frac{6}{(5-c)^4} \le \frac{6}{(5-3)^4} = \frac{6}{2^4} = \frac{3}{8}$. So

$$|R_2 f(3)| = \frac{f'''(c)}{3!}3^3 \le \frac{3}{8 \cdot 3!}3^3 = \frac{27}{16}$$