Name⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽⎽

MATH 172 Honors   Exam 1   Spring 2020

Sections 200   Solutions   P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. Compute $\displaystyle\int_0^{\pi/2} \cos^{3/2}x \sin x\, dx$.

   a. 5

   b. $\dfrac{5}{2}$

   c. $\dfrac{4}{5}$

   d. $\dfrac{2}{5}$ correct choice

   e. $\dfrac{1}{5}$

Solution: $u = \cos x$ $du = -\sin x\, dx$.

$$\int_0^{\pi/2} \cos^{3/2}x \sin x\, dx = -\int_1^0 u^{3/2}\, du = -\left[\frac{2u^{5/2}}{5}\right]_1^0 = 0 - -\frac{2}{5} = \frac{2}{5}$$

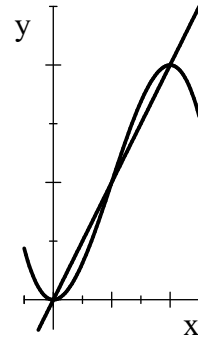**2**.   Find the total area between $y = 3x^2 - x^3$ and $y = 2x$.

   a. $\dfrac{1}{4}$

   b. $\dfrac{1}{2}$ correct choice

   c. 1

   d. 2

   e. 0

Solution: The curves intersect when $3x^2 - x^3 = 2x$ or

$0 = x^3 - 3x^2 + 2x = x(x^2 - 3x + 2) = x(x-1)(x-2)$ or $x = 0,1,2$.   So

$$A = \int_0^1 (2x - 3x^2 + x^3)\, dx + \int_1^2 (3x^2 - x^3 - 2x)\, dx = \left[x^2 - x^3 + \frac{x^4}{4}\right]_0^1 + \left[x^3 - \frac{x^4}{4} - x^2\right]_1^2$$

$$= \left(1 - 1 + \frac{1}{4}\right) + (8 - 4 - 4) - \left(1 - \frac{1}{4} - 1\right) = \frac{1}{2}$$

1-14	/56	16	/18
15	/17	17	/15
		Total	/106

1

3. Find the area between $x = y + \sin y$ and $x = y$.

a. 2   correct choice

b. 3

c. 4

d. $\dfrac{\pi}{2}$

e. π
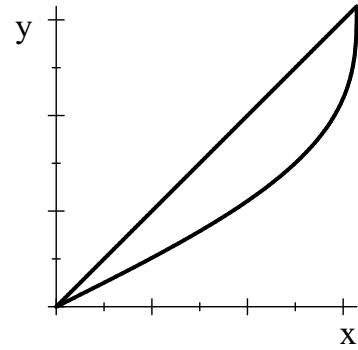
Solution: The curves intersect when $y + \sin y = y$ or $\sin y = 0$ or $y = 0, \pi$.

$$A = \int_0^\pi (y + \sin y) - (y) \, dy = [-\cos y]_0^\pi = (--1) - (-1) = 2$$

4. Find the area between $y = 3x\sqrt{16 + x^2}$ and the x-axis for $0 \le x \le 3$

a. 244

b. 54

c. $3^{3/2}$

d. 9

e. 61   correct choice

Solution: $A = \int_0^3 3x\sqrt{16 + x^2} \, dx$ We substitute $u = 16 + x^2$ and $du = 2x \, dx$ and $\dfrac{1}{2} du = x \, dx$.

$$A = \frac{3}{2} \int_{16}^{25} \sqrt{u} \, du = \left[u^{3/2} \right]_{16}^{25} = 125 - 64 = 61$$

5. Compute $\displaystyle\int_0^1 2xe^{2x} \, dx$.

a. $\dfrac{3}{2} e^2$

b. $\dfrac{1}{2} e^2$

c. $\dfrac{1}{2}(e^2 + 1)$ correct choice

d. $\dfrac{1}{2}(e^2 - 1)$

e. $\dfrac{1}{2}(3e^2 - 1)$

Solution: Use parts with
$$\begin{array}{ll} u = x & dv = 2e^{2x} \, dx \\ du = dx & v = e^{2x} \end{array}$$

$$I = xe^{2x} - \int e^{2x} \, dx = \left[xe^{2x} - \frac{1}{2}e^{2x} \right]_0^1 = \left(e^2 - \frac{1}{2}e^2 \right) - \left(-\frac{1}{2} \right) = \frac{1}{2}e^2 + \frac{1}{2}$$

2

6. Find the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

    a. 2

    b. π

    c. 2π

    d. $\frac{2}{\pi}$ correct choice

    e. $\frac{1}{\pi}$

Solution: $f_{ave} = \frac{1}{\pi} \int_0^\pi \sin x\, dx = \frac{1}{\pi}\left[-\cos x\right]_0^\pi = \frac{1}{\pi}(--1--1) = \frac{2}{\pi}$

7. Find the length of the parametric curve $x = \theta$ and $y = \ln(\cos\theta)$ for $0 \le \theta \le \frac{\pi}{4}$.

    a. $\ln\left|\sqrt{2} - 1\right|$

    b. $\ln\left|\sqrt{2} + 1\right|$ correct choice

    c. $\ln\left|\sqrt{2} + 1\right| + 1$

    d. $\ln\left|\sqrt{2} - 1\right| + 1$

    e. $\ln\left|\sqrt{2} + 1\right| - 1$

Solution: $\dfrac{dx}{d\theta} = 1$ $\dfrac{dy}{d\theta} = \dfrac{-\sin\theta}{\cos\theta} = -\tan\theta$

$L = \int_0^{\pi/4} \sqrt{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{\pi/4} \sqrt{1 + \tan^2\theta}\, d\theta = \int_0^{\pi/4} \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta|\Big|_0^{\pi/4}$

$= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0| = \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0| = \ln\left|\sqrt{2} + 1\right|$

8. The curve $(x, y) = \left(t + 1, \frac{t^2}{2} + t\right)$ for $0 \le t \le 1$ is rotated about the y-axis. Find the surface area.

    a. $\dfrac{2\pi}{3}$

    b. $\dfrac{8\pi}{3} 5^{3/2}$

    c. $\dfrac{8\pi}{3}(5^{3/2} - 1)$

    d. $\dfrac{8\pi}{3}(5^{3/2} - 2^{3/2})$

    e. $\dfrac{2\pi}{3}(5^{3/2} - 2^{3/2})$ correct choice

Solution: $\dfrac{dx}{dt} = 1$ $\dfrac{dy}{dt} = t + 1$. The radius is $r = x = t + 1$. So the surface area is:

$A = \int 2\pi r \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt = \int_0^1 2\pi(t + 1)\sqrt{1 + (t + 1)^2}\, dt = \pi\int_0^1 (2t + 2)\sqrt{t^2 + 2t + 2}\, dt$

Let $u = t^2 + 2t + 2$. Then $du = (2t + 2)\, dt$ So

$A = \pi\int_2^5 \sqrt{u}\, du = \pi\dfrac{2u^{3/2}}{3}\Big|_2^5 = \dfrac{2\pi}{3}(5^{3/2} - 2^{3/2})$

3

9. Compute $\int_1^2 x^3 \ln x \, dx$.

    a. $4\ln 2 - \dfrac{15}{16}$ correct choice

    b. $4\ln 2 - \dfrac{17}{16}$

    c. $2\ln 2 - \dfrac{7}{8}$

    d. $2\ln 2 - \dfrac{9}{8}$

    e. $2\ln 2 - \dfrac{15}{16}$

Solution: Parts with $\begin{array}{ll} u = \ln x & dv = x^3 \, dx \\ du = \frac{1}{x} \, dx & v = \frac{1}{4} x^4 \end{array}$. Then

$$I = \left[\tfrac{1}{4} x^4 \ln x - \tfrac{1}{4} \int x^4 \tfrac{1}{x} \, dx \right]_1^2 = \left[\tfrac{1}{4} x^4 \ln x - \tfrac{1}{16} x^4 \right]_1^2 = (4\ln 2 - 1) - \left(\tfrac{1}{4} \ln 1 - \tfrac{1}{16} \right) = 4\ln 2 - \tfrac{15}{16}$$

10. Compute $\int_{\pi/4}^{\pi/3} \tan^3 \theta \sec^2 \theta \, d\theta$.

    a. $\dfrac{1}{4}$

    b. $\dfrac{1}{2}$

    c. 2 correct choice

    d. 4

    e. 20

Solution: Let $u = \tan \theta$. Then $du = \sec^2 \theta \, d\theta$. So

$$\int_{\pi/4}^{\pi/3} \tan^3 \theta \sec^2 \theta \, d\theta = \int_1^{\sqrt{3}} u^3 \, du = \left[\tfrac{u^4}{4} \right]_1^{\sqrt{3}} = \left(\tfrac{9}{4} - \tfrac{1}{4} \right) = 2$$

11. Compute $\int_0^{\pi} \cos^4 \theta \, d\theta$.

    a. $\dfrac{5\pi}{8}$

    b. $\dfrac{3\pi}{8}$ correct choice

    c. $\dfrac{3\pi}{4}$

    d. $\dfrac{3\pi}{16}$

    e. $\dfrac{5\pi}{16}$

Solution: Use $\cos^2 A = \dfrac{1 + \cos 2A}{2}$.

$$\int_0^{\pi} \cos^4 \theta \, d\theta = \int_0^{\pi} \left(\frac{1 + \cos 2\theta}{2} \right)^2 d\theta = \frac{1}{4} \int_0^{\pi} (1 + 2\cos 2\theta + \cos^2 2\theta) \, d\theta = \frac{1}{4} \int_0^{\pi} \left(1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2} \right) d\theta$$

$$= \frac{1}{4} \left[\theta + \sin 2\theta + \frac{1}{2} \left(\theta + \frac{\sin 4\theta}{4} \right) \right]_0^{\pi} = \frac{1}{4} \left(\pi + \frac{\pi}{2} \right) = \frac{3\pi}{8}$$

12. Compute $\int \dfrac{1}{(4-x^2)^{3/2}}\,dx$

a. $\dfrac{\sqrt{4-x^2}}{4x} + C$

b. $\dfrac{\sqrt{4-x^2}}{2x} + C$

c. $\dfrac{1}{4\sqrt{4-x^2}} + C$

d. $\dfrac{x}{4\sqrt{4-x^2}} + C$ correct choice

e. $\dfrac{1}{2\sqrt{4-x^2}} + C$

Solution: Let $x = 2\sin\theta$. Then $dx = 2\cos\theta\,d\theta$. So

$$I = \int \frac{1}{(4-x^2)^{3/2}}\,dx = \int \frac{1}{(4-4\sin^2\theta)^{3/2}}\,2\cos\theta\,d\theta = \frac{1}{4}\int \frac{1}{(\cos^2\theta)^{3/2}}\cos\theta\,d\theta = \frac{1}{4}\int \frac{1}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{1}{4}\tan\theta + C$$

Draw a triangle with opposite side x, hypotenuse 2 and adjacent side $\sqrt{4-x^2}$. So
$$I = \frac{x}{4\sqrt{4-x^2}} + C$$

13. Compute $\int 9x^2 \cos(3x)\,dx$.

a. $3x^2\sin(3x) + 2x\cos(3x) - \frac{2}{3}\sin(3x) + C$ correct choice

b. $3x^2\sin(3x) - 2x\cos(3x) + \frac{2}{3}\sin(3x) + C$

c. $3x^2\sin(3x) + 2x\cos(3x) + \frac{2}{3}\sin(3x) + C$

d. $3x^2\sin(3x) - 2x\cos(3x) - \frac{2}{3}\sin(3x) + C$

Solution: Use parts with
$$\begin{array}{ll} u = x^2 & dv = 9\cos(3x)\,dx \\ du = 2x\,dx & v = 3\sin(3x) \end{array} \qquad I = 3x^2\sin(3x) - \int 6x\sin(3x)\,dx$$

Now use parts with
$$\begin{array}{ll} u = x & dv = 6\sin(3x)\,dx \\ du = dx & v = -2\cos(3x) \end{array} \qquad I = 3x^2\sin(3x) - \left[-2x\cos(3x) + 2\int\cos(3x)\,dx\right]$$

$$I = 3x^2\sin(3x) + 2x\cos(3x) - \frac{2}{3}\sin(3x) + C$$

14. Compute $\displaystyle\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$.

a. $\dfrac{1}{8}\sqrt{x^2-4}+C$

b. $\dfrac{x}{4}\sqrt{x^2-4}+C$

c. $\dfrac{1}{8x}\sqrt{x^2-4}+C$

d. $\dfrac{x}{8}\sqrt{x^2-4}+C$

e. $\dfrac{1}{4x}\sqrt{x^2-4}+C$ correct choice

Solution: Let $x = 2\sec\theta$. Then $dx = 2\sec\theta\tan\theta\,d\theta$. So

$$I = \int \frac{1}{x^2\sqrt{x^2-4}}\,dx = \int \frac{1}{4\sec^2\theta\sqrt{4\sec^2\theta-4}}\,2\sec\theta\tan\theta\,d\theta = \frac{1}{4}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{4}\int \cos\theta\,d\theta$$

$$= \frac{1}{4}\sin\theta$$

Draw a triangle with hypotenous x, adjacent side 2 and opposite side $\sqrt{x^2-4}$. Then

$$I = \frac{1}{4}\frac{\sqrt{x^2-4}}{x}+C$$

Work Out: (Points indicated. Part credit possible. Show all work.)

15. $\left(17\text{ points}\right)$ A bar between $x = 1$ and $x = 9$ has linear density $\delta = \dfrac{1}{\sqrt{x}}$ g/cm.

a. Find the total mass of the bar.

Solution: $M = \displaystyle\int \delta\,dx = \int_1^9 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^9 = 2(3-1) = 4$

b. Find the center of mass of the bar.

Solution: $M_1 = \displaystyle\int x\delta\,dx = \int_1^9 \frac{x}{\sqrt{x}}\,dx = \int_1^9 \sqrt{x}\,dx = \left[\frac{2x^{3/2}}{3}\right]_1^9 = \frac{2}{3}(27-1) = \frac{52}{3}$

$\bar{x} = \dfrac{M_1}{M} = \dfrac{52}{3}\dfrac{1}{4} = \dfrac{13}{3}$

16. (18 points) Compute $\int e^{4x} \cos 3x\, dx$.

Solution: Use parts with
$$\begin{aligned} u &= \cos 3x & dv &= e^{4x}\, dx \\ du &= -3\sin 3x\, dx & v &= \tfrac{1}{4}e^{4x} \end{aligned}$$
. Then

$$I = \int e^{4x}\cos 3x\, dx = \tfrac{1}{4}e^{4x}\cos 3x + \tfrac{3}{4}\int e^{4x}\sin 3x\, dx$$

Next use parts with
$$\begin{aligned} u &= \sin 3x & dv &= e^{4x}\, dx \\ du &= 3\cos 3x\, dx & v &= \tfrac{1}{4}e^{4x} \end{aligned}$$

$$I = \tfrac{1}{4}e^{4x}\cos 3x + \tfrac{3}{4}\left[\tfrac{1}{4}e^{4x}\sin 3x - \tfrac{3}{4}\int e^{4x}\cos 3x\, dx\right] = \tfrac{1}{4}e^{4x}\cos 3x + \tfrac{3}{16}e^{4x}\sin 3x - \tfrac{9}{16}I$$

$$I + \tfrac{9}{16}I = \tfrac{1}{4}e^{4x}\cos 3x + \tfrac{3}{16}e^{4x}\sin 3x$$

$$I = \tfrac{16}{25}\left(\tfrac{1}{4}e^{4x}\cos 3x + \tfrac{3}{16}e^{4x}\sin 3x\right) + C = \tfrac{4}{25}e^{4x}\cos 3x + \tfrac{3}{25}e^{4x}\sin 3x + C$$

17. (15 points) A rocket takes off from rest (initial vertical velocity $V(0) = 0$) at the ground (initial height $y(0) = 0$) and has vertical acceleration $A = 40t\cos t$. Find its height at $t = \tfrac{\pi}{2}$.

HINT: Don't get confused between the rocket velocity V and the integration by parts v.

Solution: $\dfrac{dV}{dt} = A = 40t\cos t \qquad V = \int 40t\cos t\, dt$

Use parts with
$$\begin{aligned} u &= 40t & dv &= \cos t\, dt \\ du &= 40\, dt & v &= \sin t \end{aligned}$$
.

$$V = 40t\sin t - 40\int \sin t\, dt = 40t\sin t + 40\cos t + C$$

$$V(0) = 40 + C = 0 \quad \Rightarrow \quad C = -40 \qquad V = 40t\sin t + 40\cos t - 40$$

$$\dfrac{dy}{dt} = V = 40t\sin t + 40\cos t - 40 \qquad y = \int(40t\sin t + 40\cos t - 40)\, dt$$

$$y = \int 40t\sin t\, dt + 40\sin t - 40t + K$$

For the first term, use parts with
$$\begin{aligned} u &= 40t & dv &= \sin t\, dt \\ du &= 40\, dt & v &= -\cos t \end{aligned}$$

$$\int 40t\sin t\, dt = -40t\cos t + 40\int \cos t\, dt = -40t\cos t + 40\sin t$$

$$y = -40t\cos t + 40\sin t + 40\sin t - 40t + K = -40t\cos t + 80\sin t - 40t + K$$

$$y(0) = K = 0 \quad \Rightarrow \quad K = 0$$

$$y = -40t\cos t + 80\sin t - 40t$$

$$y\left(\tfrac{\pi}{2}\right) = -40\tfrac{\pi}{2}\cos\tfrac{\pi}{2} + 80\sin\tfrac{\pi}{2} - 40\tfrac{\pi}{2} = 80 - 20\pi$$