Name____________________________________

MATH 172H Exam 2 Spring 2021

Sections 200 Solutions P. Yasskin

1-11	/59	13	/15
12	/15	14	/15
		Total	/104

Multiple Choice and Short Answer: (Points indicated.)

1. (5 pts) How many terms are there in the general partial fraction expansion of

$$\frac{6+7x}{(x-2)^2(x^2-4)(x^2+4)}?$$

Note: $\dfrac{A}{(x-2)^2}$ and $\dfrac{Bx+C}{x^2+4}$ each count as 1 term.

The number of terms is

Answer: $n =$ ___5___

Solution: We factor the denominator:

$$\frac{6+7x}{(x-2)^2(x^4-16)} = \frac{6+7x}{(x-2)^2(x-2)(x+2)(x^2+4)} = \frac{6+7x}{(x-2)^3(x+2)(x^2+4)}$$

There is 1 term for $(x+2)$, and 3 terms for $(x-2)^3$, and 1 term for (x^2+4). Or **5** terms:

$$\frac{6+7x}{(x-2)^3(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{4(x-2)^2} + \frac{D}{(x-2)^3} + \frac{Ex+F}{x^2+4}$$

2. (5 pts) Find the coeficients in the partial fraction decomposition

$$\frac{x-1}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}$$

Then compute $A-2B$.

Answer: $A-2B =$ ___4___

Solution: Clear the denominator and plug in 3 and 2:

$$x-1 = A(x-2)+B(x-3)$$

$x=3$: $\quad 2 = A(1) \quad A = 2$ $\qquad$ $x=2$: $\quad 1 = B(-1) \quad B=-1$

$$\frac{x-1}{x^2-5x+6} = \frac{2}{x-3} + \frac{-1}{x-2} \qquad A-2B = (2)-2(-1) = 4$$

3. (5 pts) Given that $\dfrac{32}{x^4-16} = \dfrac{1}{x-2} - \dfrac{1}{x+2} - \dfrac{4}{x^2+4}$ compute $\displaystyle\int_0^1 \frac{32}{x^4-16}\,dx$.

a. $-\ln 3 - \arctan \frac{1}{2}$

b. $-\ln 3 - 2\arctan \frac{1}{2}$ correct choice

c. $\ln 2 - \ln 3 - \arctan \frac{1}{2}$

d. $\ln 2 - \ln 3 - 2\arctan \frac{1}{2}$

e. $2\ln 2 - \ln 3 - \arctan \frac{1}{2}$

f. $2\ln 2 - \ln 3 - 2\arctan \frac{1}{2}$

Solution: $\displaystyle\int \frac{32}{x^4-16}\,dx = \int \frac{1}{x-2} - \frac{1}{x+2} - \frac{4}{x^2+4}\,dx$

On the last term we make the substitution $x = 2\tan\theta \qquad dx = 2\sec^2\theta\,d\theta$.

$$\int \frac{4}{x^2+4}\,dx = \int \frac{4}{4\tan^2\theta+4} 2\sec^2\theta\,d\theta = 2\int 1\,d\theta = 2\theta + C = 2\arctan\frac{x}{2} + C$$

So $\displaystyle\int_0^1 \frac{32}{x^4-16}\,dx = \left[\ln|x-2| - \ln|x+2| - 2\arctan\frac{x}{2}\right]_0^1 = -\ln 3 - 2\arctan\frac{1}{2}$

4. (5 pts) The region between $x = 25 - y^2$ and the y-axis is rotated about the y-axis. Find the volume.

a. $\dfrac{2^4 5^4}{3}\pi$ correct choice

b. $\dfrac{2^4 5^3}{3}\pi$

c. $\dfrac{2^3 5^4}{3}\pi$

d. $2^3 5^5 3\pi$

e. $2^2 5^4 3\pi$

Solution: This is a y-integral, the slices are horizontal and rotate into disks. The radius is $r = x = 25 - y^2$. So the volume is:

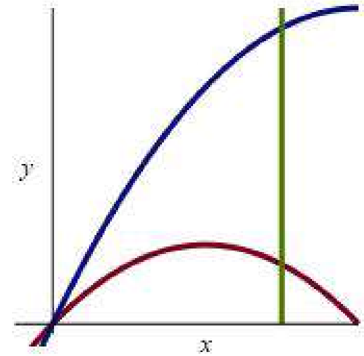
$$V = \int_{-5}^{5} \pi r^2\,dy = \int_{-5}^{5} \pi(25-y^2)^2\,dy = \int_{-5}^{5} \pi(25^2 - 50y^2 + y^4)\,dy = \pi\left[25^2 y - \frac{50}{3}y^3 + \frac{y^5}{5}\right]_{-5}^{5}$$

$$= 2\pi\left(5^5 - \frac{2}{3}5^5 + \frac{5^5}{5}\right) = 2\pi 5^5\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 2\pi 5^5 \frac{15-10+3}{15} = \frac{10^4}{3}\pi$$

5. (5 pts) The base of a solid is the region bounded by

$y = 4x - x^2$ and $y = 8x - x^2$ and $x = 3$.

The slices perpendicular to the x-axis are semicircles with a diameter on the base. Find the volume.



a. 9π
b. 12π
c. 18π correct choice
d. 24π
e. 36π
f. 48π
g. 72π
h. 96π
i. 150π
j. 210π
k. 270π
l. 360π

Solution: The diameter of each semicircle is $d = (8x - x^2) - (4x - x^2) = 4x$. Then the radius is $r = 2x$. So the area of each semicircle is $A(x) = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi 4x^2 = 2\pi x^2$. And the volume is

$$V = \int_0^3 A(x)\,dx = \int_0^3 2\pi x^2\,dx = 2\pi\left[\frac{x^3}{3}\right]_0^3 = 18\pi$$

6. (5 pts) The region bounded by $y = 4x - x^2$ and $y = 8x - x^2$ and $x = 3$ (See figure above.) is rotated about the x-axis. Find the volume.

a. 9π
b. 12π
c. 18π
d. 24π
e. 36π
f. 48π
g. 72π
h. 96π
i. 150π
j. 210π
k. 270π correct choice
l. 360π

Solution: The slices are vertical and rotate into washers. The outer radius is $R = 8x - x^2$. The inner radius is $r = 4x - x^2$. So the volume is

$$V = \int_0^3 \pi(R^2 - r^2)\,dx = \int_0^3 \pi\left((8x - x^2)^2 - (4x - x^2)^2\right)dx = \int_0^3 \pi((64x^2 - 16x^3 + x^4) - (16x^2 - 8x^3 + x^4))dx$$

$$= \int_0^3 \pi(48x^2 - 8x^3)dx = \pi\left[16x^3 - 2x^4\right]_0^3 = \pi(16 \cdot 3^3 - 2 \cdot 3^4) = 27\pi(16 - 6) = 270\pi$$

7. (5 pts) The region bounded by $y = 4x - x^2$ and $y = 8x - x^2$ and $x = 3$ (See figure above.) is rotated about the y-axis. Find the volume.

a. 9π
b. 12π
c. 18π
d. 24π
e. 36π
f. 48π
g. 72π correct choice
h. 96π
i. 150π
j. 210π
k. 270π
l. 360π

Solution: The slices are vertical and rotate into cylinders. The radius is $r = x$ and the height is $h = (8x - x^2) - (4x - x^2) = 4x$. So the volume is

$$V = \int_0^3 2\pi rh\,dx = 2\pi\int_0^3 (x)(4x)dx = 8\pi\left[\frac{x^3}{3}\right]_0^3 = 72\pi$$

8. (5 pts) Compute the improper integral $\int_1^\infty xe^{-x}\,dx$.

a. 0

b. $\frac{1}{e}$

c. $\frac{2}{e}$ correct choice

d. $\frac{4}{e}$

e. ∞

Solution: We use integration by parts with $\begin{array}{ll} u = x & dv = e^{-x}\,dx \\ du = dx & v = -e^{-x} \end{array}$:

$$\int_1^\infty xe^{-x}\,dx = \left[-xe^{-x} + \int e^{-x}\,dx\right]_1^\infty = \left[-xe^{-x} - e^{-x}\right]_1^\infty = 0 - (-e^{-1} - e^{-1}) = \frac{2}{e}$$

9. (5 pts) Compute the improper integral $\int_0^1 \frac{2}{\sqrt{1-x^2}}\,dx$.

a. π correct choice

b. $\frac{\pi}{2}$

c. $\frac{\pi}{3}$

d. $\frac{\pi}{4}$

e. divergent

Solution: You can use the trig substitution $x = \sin\theta$, or simply remember the antiderivative:

$$\int_0^1 \frac{2}{\sqrt{1-x^2}}\,dx = \left[2\arcsin x\right]_0^1 = 2\arcsin 1 - 2\arcsin 0 = 2\left(\frac{\pi}{2}\right) = \pi$$

10. (5 pts) Compute the improper integral $\int_0^{16} \frac{1}{(x-8)^{4/3}}\,dx$.

a. 0

b. $-\frac{3}{4}$

c. $-\frac{3}{2}$

d. -3

e. divergent correct choice

Solution: $\int_0^{16} \frac{1}{(x-8)^{4/3}}\,dx = \int_0^{8} \frac{1}{(x-8)^{4/3}}\,dx + \int_8^{16} \frac{1}{(x-8)^{4/3}}\,dx$

$$\int_0^{8} \frac{1}{(x-8)^{4/3}}\,dx = \lim_{b\to 8^-}\left[\frac{-3}{(x-8)^{1/3}}\right]_0^b = \frac{-3}{0^-} - \frac{-3}{(-8)^{1/3}} = \infty - \frac{3}{2} = \infty$$

Since this half is divergent, the whole integral is divergent.

11. (9 pts) The rest position of a certain spring is at $x = 0$ cm. It takes 72 ergs of work to stretch it from $x = 4$ cm to $x = 8$ cm.

a. Find the spring constant.

$k =$ _____ $\dfrac{\text{dynes}}{\text{cm}}$

Solution: $W = \int_4^8 kx\,dx = \left[k\frac{x^2}{2}\right]_4^8 = k(32-8) = 24k = 72 \qquad k = 3\ \dfrac{\text{dynes}}{\text{cm}}$

b. How much work does it take to stretch it from $x = 2$ cm to $x = 6$ cm?

$W =$ _____ ergs

Solution: $F = kx = 3x \qquad W = \int_2^6 3x\,dx = \left[3\frac{x^2}{2}\right]_2^6 = 3(18-2) = 32$ ergs

c. How much forch is needed to hold it at $x = 5$ cm?

$F =$ _____ dynes

Solution: $F = 3x = 3 \bullet 5 = 15$ dynes

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (15 pts) Find the partial fraction expansion for $\dfrac{2x+9}{x^3+9x} = \dfrac{1}{x} + \dfrac{-x+2}{x^2+9}$.

(3 pts Extra Credit for a complex number solution.)

$A ==$ _____ $\qquad B =$ _____ $\qquad C =$ _____

Solution: We factor the denominator, write the general partial fraction expansion and clear the denominator:

$$\frac{2x+9}{x^3+9x} = \frac{2x+9}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$2x+9 = A(x^2+9) + (Bx+C)(x)$$

We plug in $x = 0, 3, -3$:

$x = 0$: $\quad 9 = A(9) \quad \Rightarrow \quad A = 1$

$x = 3$: $\quad 6+9 = A(9+9) + (B3+C)(3) \quad \Rightarrow \quad 15 = 18 + 3(3B+C) \quad \Rightarrow \quad 3B + C = -1$

$x = -3$: $\quad -6+9 = A(9+9) + (-B3+C)(-3) \quad \Rightarrow \quad 3 = 18 - 3(-3B+C) \quad \Rightarrow \quad -3B + C = 5$

Adding: $\quad 2C = 4 \quad \Rightarrow \quad C = 2$

Subtracting: $\quad 6B = -6 \quad \Rightarrow \quad B = -1$

So: $\quad \dfrac{2x+9}{x^3+9x} = \dfrac{1}{x} + \dfrac{-x+2}{x^2+9}$

Complex Solution: Instead of plugging in $x = 3, -3$, we plug in $x = 3i$:

$6i + 9 = A(-9+9) + (3iB+C)(3i) = -9B + 3iC \qquad 9 = -9B \qquad 6i = 3iC \qquad B = -1 \qquad C = 2$

13. (15 pts) Determine if the improper integral $\int_2^\infty \frac{2}{e^x + x}\,dx$ converges or diverges. Do the integral exactly or use a Comparison Test. If you do the integral exactly, be sure to state all substitutions you make and their differentials. If you use a comparison, be sure to state the comparison integral, explain why the comparison integral converges or diverges and check the inequality.
(You will be graded for good sentences!)

___X__Convergent _____Divergent

Solution: For large x, e^x is much larger then x. So to construct a comparison integral, we keep the e^x and throw away the x. So our comparison inegral and its value is

$$\int_2^\infty \frac{2}{e^x}\,dx = \int_2^\infty 2e^{-x}\,dx = \left[-2e^{-x}\right]_2^\infty = 0 - -2e^{-2} = \frac{2}{e^2}$$

which is finite (convergent). Now $e^x + x > e^x$. So $\frac{2}{e^x + x} < \frac{2}{e^x}$. Therefore

$$\int_2^\infty \frac{2}{e^x + x}\,dx < \int_2^\infty \frac{2}{e^x}\,dx$$

Since the larger integral is finite (convergent), so is the smaller integral.

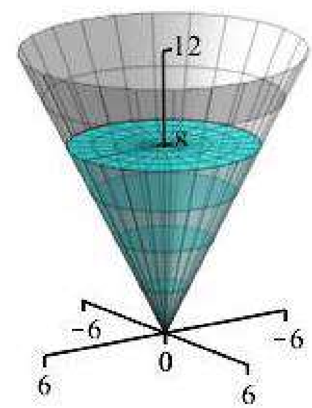
14. (15 pts) A cone is 12 cm tall and 6 cm in radius at the top.

It is filled with salt water of density $\delta = 1.02\ \frac{\text{gm}}{\text{cm}^3}$ to a depth of 8 cm.

Find the work done to pump all the water over the top of the cone.

For numerical computations, use the approximation that

$\delta g = 9.8 \cdot 1.02 \approx 10\ \frac{\text{gm} \cdot \text{cm}}{\text{sec}^2}$.



Solution: The slice at height y is a disk of radius r. By similar triangles, $\frac{r}{y} = \frac{6}{12}$ or $r = \frac{1}{2}y$. So the volume of a slice is $dV = \pi r^2\,dy = \frac{\pi}{4}y^2\,dy$ and its weight is $dF = \delta g\,dV = 10\frac{\pi}{4}y^2\,dy$. This slice is lifted a distance $D = 12 - y$. There is water between $y = 0$ and $y = 8$, which are the limits of integration. So the work done is:

$$W = \int_0^8 D\,dF = \int_0^8 (12 - y)\,10\frac{\pi}{4}y^2\,dy = 5\pi \int_0^8 \left(6y^2 - \frac{1}{2}y^3\right) dy$$

$$= 5\pi\left[2y^3 - \frac{y^4}{8}\right]_0^8 = 5\pi[2 \cdot 8^3 - 8^3] = 5 \cdot 8^3\pi = 2560\pi$$