Name____________________________________

1-10	/50	13	/15
11	/15	14	/15
12	/10	Total	/105

MATH 172H Final Spring 2021

Sections 200 Solutions P. Yasskin

Anything above 100 is extra credit.

Multiple Choice and Short Answer: (5 Points Each)

1. Compute $\int_0^1 2x\arctan x\,dx$. HINT: $\dfrac{a}{1+a} = \dfrac{1+a}{1+a} - \dfrac{1}{1+a}$

a. $\frac{\pi}{4}+1$
d. $\frac{\pi}{4}$
g. $\frac{\pi}{4}-1$

b. $\frac{\pi}{2}+1$
e. $\frac{\pi}{2}$
h. $\frac{\pi}{2}-1$ correct choice

c. $\pi+1$
f. π
i. $\pi-1$

Solution: Integrate by parts $u = \arctan x \quad dv = 2x\,dx$, $du = \dfrac{1}{1+x^2}\,dx \quad v = x^2$

$$\int_0^1 2x\arctan x\,dx = \left[x^2\arctan x\right]_0^1 - \int_0^1 \frac{x^2}{1+x^2}\,dx = \left[x^2\arctan x\right]_0^1 - \int_0^1 1 - \frac{1}{1+x^2}\,dx$$
$$= \left[x^2\arctan x - x + \arctan x\right]_0^1 = \left[\arctan 1 - 1 + \arctan 1\right] - 0 = 2\frac{\pi}{4} - 1 = \frac{\pi}{2} - 1$$

2. Compute $\int_0^{\pi/4} \tan x\sec^4 x\,dx$.

a. $-\frac{1}{4}$

b. 0

c. $\frac{1}{4}$

d. $\frac{3}{4}$ correct choice

e. $\frac{5}{4}$

Solution: We use the substitution $u = \sec x$ and $du = \sec x\tan x\,dx$:

$$\int_0^{\pi/4} \tan x\sec^4 x\,dx = \int_1^{\sqrt{2}} u^3\,du = \left[\frac{u^4}{4}\right]_1^{\sqrt{2}} = \frac{\sqrt{2}^{\,4}}{4} - \frac{1}{4} = 1 - \frac{1}{4} = \frac{3}{4}$$

Or we can use the substitution $u = \tan x$ and $du = \sec^2 x\,dx$ with the identity $\sec^2 x = \tan^2 x + 1 = u^2 + 1$:

$$\int_0^{\pi/4} \tan x\sec^4 x\,dx = \int_0^1 u(u^2+1)\,du = \left[\frac{u^4}{4} + \frac{u^2}{2}\right]_0^1 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

3. Compute $\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$.

a. $\frac{1}{2}\operatorname{arcsec}\frac{x}{2}+C$

b. $\frac{1}{4}\operatorname{arcsec}\frac{x}{4}+C$

c. $\frac{x}{2}\operatorname{arcsec}\frac{x}{2}+C$

d. $\frac{x}{4\sqrt{x^2-4}}+C$

e. $\frac{\sqrt{x^2-4}}{4x}+C$ correct choice

Solution: $x = 2\sec\theta \qquad dx = 2\sec\theta\tan\theta\,d\theta$

$$\int \frac{1}{x^2\sqrt{x^2-4}}\,dx = \int \frac{2\sec\theta\tan\theta\,d\theta}{4\sec^2\theta\sqrt{4\sec^2\theta-4}} = \frac{1}{4}\int \frac{\tan\theta\,d\theta}{\sec\theta\sqrt{\tan^2\theta}} = \frac{1}{4}\int\cos\theta\,d\theta = \frac{1}{4}\sin\theta + C$$

Since $\sec\theta = \frac{x}{2}$, draw a triangle with hypotenuse x and adjacent side 2. Then the opposite side is $\sqrt{x^2-4}$ and so $\sin\theta = \frac{\sqrt{x^2-4}}{x}$. Therefore:

$$\int \frac{1}{x^2\sqrt{x^2-4}}\,dx = \frac{\sqrt{x^2-4}}{4x} + C$$

Check: $\frac{d}{dx}\frac{\sqrt{x^2-4}}{4x} = \frac{4x\frac{x}{\sqrt{x^2-4}} - 4\sqrt{x^2-4}}{16x^2} = \frac{4x^2-4(x^2-4)}{16x^2\sqrt{x^2-4}} = \frac{1}{x^2\sqrt{x^2-4}}$

4. Find the center of mass of a 2 m bar whose density is $\delta = \frac{1}{x^4}$ for $2 \le x \le 4$.

a. $\frac{9}{7}$

b. $\frac{18}{7}$ correct choice

c. $\frac{18}{9}$

d. $\frac{3}{32}$

e. $\frac{3}{16}$

Solution: $M = \int_2^4 \delta\,dx = \int_2^4 \frac{1}{x^4}\,dx = \left[\frac{-1}{3x^3}\right]_2^4 = -\frac{1}{192} + \frac{1}{24} = \frac{8-1}{192} = \frac{7}{192}$

$M_1 = \int_2^4 x\delta\,dx = \int_2^4 \frac{1}{x^3}\,dx = \left[\frac{-1}{2x^2}\right]_2^4 = -\frac{1}{32} + \frac{1}{8} = \frac{4-1}{32} = \frac{3}{32} \qquad \bar{x} = \frac{M_1}{M} = \frac{3}{32}\frac{192}{7} = \frac{18}{7}$

5. Find the arc length of the parametric curve $\vec{r}(t) = \left(t^2, \frac{2}{3}t^3\right)$ for $0 \le t \le \sqrt{3}$.

a. 6

b. $\frac{16}{3}$

c. $\frac{14}{3}$ correct choice

d. 4

e. $\frac{8}{3}$

Solution: $\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = 2t^2$

$$L = \int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^{\sqrt{3}} \sqrt{(2t)^2 + (2t^2)^2}\,dt = \int_0^{\sqrt{3}} 2t\sqrt{1+t^2}\,dt = \left[2\frac{(1+t^2)^{3/2}}{3} \right]_0^{\sqrt{3}}$$
$$= \frac{2(1+3)^{3/2}}{3} - \frac{2(1)^{3/2}}{3} = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}$$

6. The region between the parabola $x = 5y - y^2$ and the y-axis is rotated about the y-axis. Find the volume swept out.

a. $V = \frac{5^4}{3}\pi$

b. $V = \frac{5^4}{6}\pi$ correct choice

c. $V = \frac{5^5}{6}\pi$

d. $V = \frac{5^5}{12}\pi$

e. $V = 3 \cdot 5^6\pi$

Solution: We do a y-integral. The rectangles are horizontal. A rectangle sweeps out a disk. The radius is $r = x = 5y - y^2$. The limits are $y = 0, 5$. So the volume is:

$$V = \int_0^5 \pi r^2 dy = \int_0^5 \pi(5y - y^2)^2\,dy = \pi\int_0^5 (25y^2 - 10y^3 + y^4)\,dy = \pi\left[25\frac{y^3}{3} - 10\frac{y^4}{4} + \frac{y^5}{5}\right]_0^5$$
$$= \pi\left(\frac{5^5}{3} - \frac{5^5}{2} + 5^4\right) = \pi 5^4\left(\frac{5}{3} - \frac{5}{2} + 1\right) = \pi 5^4\left(\frac{5}{3} - \frac{5}{2} + 1\right) = \frac{5^4}{6}\pi$$

7. Solve the initial value problem

$$t^4 \frac{dy}{dt} - t^3 y = t^2 \qquad \text{with} \quad y(1) = \frac{3}{2}$$

Then $y(2) =$

a. 30

b. 15

c. $\frac{15}{2}$

d. $\frac{15}{4}$ correct choice

e. $\frac{15}{8}$

Solution: The equation is linear. The standard form is

$$\frac{dy}{dt} - \frac{1}{t} y = \frac{1}{t^2}$$

So $P = -\frac{1}{t}$ and the integrating factor is $I = \exp\left(-\int \frac{1}{t} dt\right) = \exp(-\ln t) = \frac{1}{t}$.

We multiply the standard equation by the integrating factor:

$$\frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2} y = \frac{1}{t^3}$$

$$\frac{d}{dt}\left(\frac{1}{t} y\right) = \frac{1}{t^3}$$

$$\frac{1}{t} y = -\frac{1}{2t^2} + C$$

$$y = -\frac{1}{2t} + Ct$$

The initial condition says $y = \frac{3}{2}$ when $t = 1$. So

$$\frac{3}{2} = -\frac{1}{2} + C \quad \Rightarrow \quad C = 2$$

$$y = -\frac{1}{2t} + 2t$$

$$y(2) = -\frac{1}{4} + 4 = \frac{15}{4}$$

8. Find a power series about $x = 0$ for $f(x) = \dfrac{2x}{(1-x^2)^2}$.

a. $\sum_{n=0}^{\infty} 2nx^{2n-1}$ correct choice

b. $\sum_{n=0}^{\infty} 2x^{2n-1}$

c. $\sum_{n=0}^{\infty} 2nx^{2n+1}$

d. $\sum_{n=0}^{\infty} 2x^{2n+1}$

e. $\sum_{n=0}^{\infty} 4n^3 x^{2n-1}$

f. $\sum_{n=0}^{\infty} 4n^3 x^{2n+1}$

Solution: $\dfrac{1}{1-x^2} = \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} x^{2n} \qquad \dfrac{d}{dx}\dfrac{1}{1-x^2} = \dfrac{-1(-2x)}{(1-x^2)^2} = \dfrac{2x}{(1-x^2)^2} = \sum_{n=0}^{\infty} 2nx^{2n-1}$

9. Compute $\lim\limits_{n\to\infty} n^{2/n}$.

a. 0

b. 1 correct choice

c. 2

d. e

e. e^2

Solution: $\lim\limits_{n\to\infty} n^{2/n} = \lim\limits_{n\to\infty} e^{\ln n^{2/n}} = e^{\lim\limits_{n\to\infty} \frac{2}{n}\ln n} \overset{l'H}{=} e^{\lim\limits_{n\to\infty} \frac{2}{n}} = e^0 = 1$

10. Compute $\lim\limits_{x\to 0} \dfrac{x^2 - \frac{x^6}{6} - \sin(x^2)}{x^{10}}$.

a. $-\dfrac{1}{120}$ correct choice

b. $-\dfrac{1}{24}$

c. 0

d. $\dfrac{1}{24}$

e. $\dfrac{1}{120}$

f. ∞

Solution: $\sin u = u - \dfrac{u^3}{6} + \dfrac{u^5}{120} - \cdots \qquad \sin x^2 = x^2 - \dfrac{x^6}{6} + \dfrac{x^{10}}{120} - \cdots$

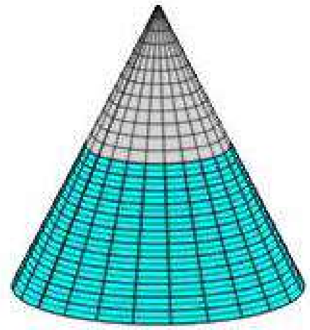
$$\lim_{x\to 0} \frac{x^2 - \frac{x^6}{6} - \sin(x^2)}{x^{10}} = \lim_{x\to 0} \frac{x^2 - \frac{x^6}{6} - \left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \cdots\right)}{x^{10}} = \lim_{x\to 0} \frac{-\frac{x^{10}}{120} + \cdots}{x^{10}} = -\frac{1}{120}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (15 points) Work Out Problem

A water tank has the shape of a cone with the vertex atthe top. Its height is $H = 16$ ft and its radius is $R = 8$ ft. It is filled with salt water to a depth of 10 ft which weighs $\delta = 64 \dfrac{\text{lb}}{\text{ft}^3}$.

Find the work done to pump the water out the top of the tank.



Solution: Put the y-axis measuring down from the top.

The slice which is a distance y down from the top is a circle of radius r.

By similar triangles, $\frac{r}{y} = \frac{R}{H} = \frac{8}{16} = \frac{1}{2}$. So $r = \frac{1}{2}y$.

The area is $A = \pi r^2 = \frac{\pi y^2}{4}$ and the volume of the slice of thickness dy is $dV = A\,dy = \frac{\pi y^2}{4}\,dy$.

It weighs $dF = \delta\,dV = 64\frac{\pi y^2}{4}\,dy = 16\pi y^2\,dy$. It is lifted a distance $D = y$.

There is water between $y = 6$ and $y = 16$. So the work done is

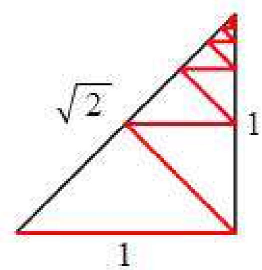
$$W = \int_6^{16} D\,dF = \int_6^{16} y\,16\pi y^2\,dy = \left[16\pi\frac{y^4}{4}\right]_6^{16} = 4\pi(16^4 - 6^4)\ \text{ft-lb}$$

12. (10 points) Work Out Problem

Find the length of the infinite zigzag within the $45°$ right triangle, shown at the right.

Each diagonal is at $45°$.

The total length includes the base.



$L =$ __$2 + \sqrt{2}$__

Solution: Each horizontal line has half the length of the previous and starts with 1.

Each diagonal line has half the length of the previous and starts with $\frac{\sqrt{2}}{2}$.

So the total length is

$$L = \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) + \frac{\sqrt{2}}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = \left(1 + \frac{\sqrt{2}}{2}\right)\frac{1}{1 - \frac{1}{2}} = 2 + \sqrt{2}$$

13. (15 points) Work Out Problem

A tropical fish tank contains 100 liters of salt water with initial salt concentration $35\,\frac{\text{grams}}{\text{liter}}$.

In order to reduce the concentration, salt water with concentration $25\,\frac{\text{grams}}{\text{liter}}$ is added at $2\,\frac{\text{liters}}{\text{minute}}$.

The water is kept thoroughly mixed and drained at $2\,\frac{\text{liters}}{\text{minute}}$.

How long will it take until the concentration is reduced to $30\,\frac{\text{grams}}{\text{liter}}$?

Let $S(t)$ be the number of grams of salt in the 100 liters of water at time t.

Then the concentration is $\frac{S(t)}{100}\,\frac{\text{grams}}{\text{liter}}$.

a. What is the initial condition? (Quantity not concentration.)

Solution: $S(0) = 35\,\frac{\text{grams}}{\text{liter}}\,100 \text{ liters} = 3500 \text{ grams}$

$S(0) =$ _____3500_____

b. What is the differential equation?

Solution: $\frac{dS}{dt} = [\text{Rate In}] - [\text{Rate Out}] = 25\,\frac{\text{grams}}{\text{liter}}\,2\,\frac{\text{liters}}{\text{minute}} - \frac{S(t)}{100}\,\frac{\text{grams}}{\text{liter}}\,2\,\frac{\text{liters}}{\text{minute}}$

$\frac{dS}{dt} =$ _____$50 - \frac{1}{50}S(t)$_____

c. Solve the initial value problem.

Solution Method 1: Linear: Standard form: $\frac{dS}{dt} + \frac{1}{50}S(t) = 50$ $\quad P = \frac{1}{50}$ $\quad I = e^{\int P dt} = e^{t/50}$

$e^{t/50}\frac{dS}{dt} + \frac{1}{50}e^{t/50}S(t) = 50e^{t/50}$ $\quad \frac{d}{dt}[e^{t/50}S(t)] = 50e^{t/50}$ $\quad e^{t/50}S(t) = 2500e^{t/50} + C$

$S(0) = 3500$ at $t = 0$ $\quad 3500 = 2500 + C$ $\quad C = 1000$ $\quad e^{t/50}S(t) = 2500e^{t/50} + 1000$

$S(t) =$ _____$2500 + 1000e^{-t/50}$_____

Solution Method 2: Separable: $\int \frac{dS}{50 - \frac{1}{50}S}dt = \int dt$ $\quad -50\ln\left|50 - \frac{1}{50}S\right| = t + C$

$\ln\left|50 - \frac{1}{50}S\right| = -\frac{t}{50} - \frac{C}{50}$ $\quad 50 - \frac{1}{50}S = \pm e^{-t/50 - C/50} = Ae^{-t/50}$ $\quad S(0) = 3500$ at $t = 0$

$50 - \frac{3500}{50} = Ae^{0}$ $\quad A = 50 - 70 = -20$ $\quad 50 - \frac{1}{50}S = -20e^{-t/50}$ $\quad 2500 - S = -1000e^{-t/50}$

$S(t) =$ _____$2500 + 1000e^{-t/50}$_____

d. Find the time when $\frac{S(t)}{100} = 30$.

Solution: $\frac{S(t)}{100} = 25 + 10e^{-t/50} = 30$ $\quad 10e^{-t/50} = 5$ $\quad e^{-t/50} = \frac{1}{2}$ $\quad -\frac{t}{50} = \ln\frac{1}{2}$ $\quad t = 50\ln 2$

$t =$ _____$50\ln 2$_____

14. (15 points) Work Out Problem

Find the interval of convergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[2]{n} - \sqrt[3]{n}} \frac{(x-3)^n}{2^n}$.

a. Find the radius of convergence.

Solution: We apply the Ratio Test:

$$|a_n| = \frac{1}{\sqrt[2]{n} - \sqrt[3]{n}} \frac{|x-3|^n}{2^n} \qquad |a_{n+1}| = \frac{1}{\sqrt[2]{n+1} - \sqrt[3]{n+1}} \frac{|x-3|^{n+1}}{2^{n+1}}$$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{\sqrt[2]{n+1} - \sqrt[3]{n+1}} \frac{|x-3|^{n+1}}{2^{n+1}} \frac{\sqrt[2]{n} - \sqrt[3]{n}}{1} \frac{2^n}{|x-3|^n}$$

$$= \frac{|x-3|}{2} \lim_{n\to\infty} \frac{\sqrt[2]{n} - \sqrt[3]{n}}{\sqrt[2]{n+1} - \sqrt[3]{n+1}} = \frac{|x-3|}{2} \lim_{n\to\infty} \frac{1 - n^{-1/6}}{1 - (n+1)^{-1/6}} = \frac{|x-3|}{2} < 1$$

$|x-3| < 2 \qquad R = 2$ Open interval: $(1,5)$

b. Check the convergence at the left endpoint.
Be sure to name any convergence test you use and check out all conditions.

Solution: $x = 1$: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[2]{n} - \sqrt[3]{n}} \frac{(-2)^n}{2^n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[2]{n} - \sqrt[3]{n}}$

This converges by the Alternating Series Test, because $b_n = \dfrac{1}{\sqrt[2]{n} - \sqrt[3]{n}}$ is positive, decreasing and $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt[2]{n} - \sqrt[3]{n}} = 0$.

c. Check the convergence at the right endpoint.
Be sure to name any convergence test you use and check out all conditions.

Solution: $x = 5$: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[2]{n} - \sqrt[3]{n}} \frac{(2)^n}{2^n} = \sum_{n=2}^{\infty} \frac{1}{\sqrt[2]{n} - \sqrt[3]{n}}$

We compare to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[2]{n}}$ which is a divergent p-series since $p = \frac{1}{2} < 1$. We compute the limit:

$\displaystyle L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[2]{n}}{\sqrt[2]{n} - \sqrt[3]{n}} = \lim_{n\to\infty} \frac{1}{1 - n^{-1/6}} = 1$ Since $0 < L < \infty$

by the Limit Comparison Test, the original series also diverges.

d. State the interval of convergence.

Solution: The interval of convergence is $[1,5)$.