Name________________________________

MATH 172 Honors Exam 2 Spring 2022

Sections 200 Solutions P. Yasskin

| | | | |
|---|---|---|---|
| 1 | /8 | 6 | /10 |
| 2 | /10 | 7 | /10 |
| 3 | /10 | 8 | /10 |
| 4 | /10 | 9 | /5 |
| 5 | /10 | 10 | /20 |
| | | Total | /103 |

Multiple Choice: (8 points. No part credit. Circle your answers.)

**1**. (8 points) Consider the general partial fraction expansion $\dfrac{x^3 - x^2}{(x^2+4)^2} = \dfrac{Ax+B}{x^2+4} + \dfrac{Cx+D}{(x^2+4)^2}$.

Find the coefficients. (Circle 1 answer in each row.)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $A =$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $\boxed{1}$ | $2$ | $3$ | $4$ |
| $B =$ | $-4$ | $-3$ | $-2$ | $\boxed{-1}$ | $0$ | $1$ | $2$ | $3$ | $4$ |
| $C =$ | $\boxed{-4}$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
| $D =$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $\boxed{4}$ |

**Solution**: We clear the denominator and expand:

$$\begin{aligned} x^3 - x^2 &= (Ax+B)(x^2+4) + (Cx+D) \\ &= Ax^3 + Bx^2 + (4A+C)x + (4B+D) \end{aligned}$$

We equate coefficients: $A = 1$ $\quad B = -1$ $\quad C = -4A = -4$ $\quad D = -4B = 4$ So:

$$\frac{x^3 - x^2}{(x^2+4)^2} = \frac{x-1}{x^2+4} + \frac{-4x+4}{(x^2+4)^2}$$

---

Work Out: (Points indicated. Part credit possible. Show all work.)

**2**. (10 points) Given the partial fraction expansion $\dfrac{2x-2}{x^4-1} = \dfrac{1}{x+1} + \dfrac{1-x}{x^2+1}$, compute $\displaystyle\int_0^1 \frac{2x-2}{x^4-1}\,dx$.
Simplify and evaluate all trig and inverse trig functions.

**Solution**: $\displaystyle\int_0^1 \frac{2x-2}{x^4-1}\,dx = \int_0^1 \frac{1}{x+1} + \frac{1-x}{x^2+1}\,dx = \int_0^1 \frac{1}{x+1} + \frac{1}{x^2+1} - \frac{x}{x^2+1}\,dx$

$$= \left[\ln|x+1| + \arctan x - \frac{1}{2}\ln|x^2+1|\right]_0^1 = \ln 2 + \arctan 1 - \frac{1}{2}\ln 2 = \frac{1}{2}\ln 2 + \frac{\pi}{4}$$

**3**. (10 points) Compute $\int_0^1 \frac{e^{-x}}{1-e^{-x}}\,dx$ or show why it diverges and whether it is $\infty$ or $-\infty$.

**Solution**: Let $u = 1 - e^{-x}$. Then $du = e^{-x}\,dx$ and

$$\int_0^1 \frac{e^{-x}}{1-e^{-x}}\,dx = \int_0^{1-e^{-1}} \frac{1}{u}\,du = [\ln|u|]_0^{1-e^{-1}} = \ln|1-e^{-1}| - \lim_{u\to0^+} \ln u = \ln|1-e^{-1}| - -\infty = \infty$$

**4**. (10 points) Show why $\int_1^\infty \frac{x+\sin x}{x^{5/2}}\,dx$ converges or diverges.

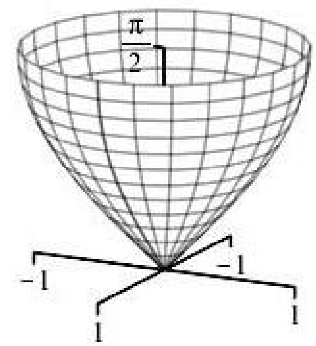
**Solution**: Notice $x-1 \le x+\sin x \le x+1$. So $\frac{x-1}{x^{5/2}} \le \frac{x+\sin x}{x^{5/2}} \le \frac{x+1}{x^{5/2}}$.

$$\int_1^\infty \frac{x-1}{x^{5/2}}\,dx = \int_1^\infty (x^{-3/2} - x^{-5/2})\,dx = \left[-2x^{-1/2} + \frac{2x^{-3/2}}{3}\right]_1^\infty = (-0+0) - \left(-2+\frac{2}{3}\right) = \frac{4}{3}$$

$$\int_1^\infty \frac{x+1}{x^{5/2}}\,dx = \int_1^\infty (x^{-3/2} + x^{-5/2})\,dx = \left[-2x^{-1/2} - \frac{2x^{-3/2}}{3}\right]_1^\infty = (-0-0) - \left(-2-\frac{2}{3}\right) = \frac{8}{3}$$

Since $\int_1^\infty \frac{x+\sin x}{x^{5/2}}\,dx \le \int_1^\infty \frac{x+1}{x^{5/2}}\,dx = \frac{8}{3}$, we know $\int_1^\infty \frac{x+\sin x}{x^{5/2}}\,dx$ converges.

**5**. (10 points) A cup is made by revolving the curve $x = \sin y$ about the $y$-axis for $0 \le y \le \frac{\pi}{2}$. Find its volume.




**Solution**: The radius is $r = x = \sin y$. The cross sectional area is $A = \pi r^2 = \pi \sin^2 y$.

The volume is $V = \int_0^{\pi/2} A(y)\,dy = \int_0^{\pi/2} \pi \sin^2 y\,dy = \int_0^{\pi/2} \pi \frac{1-\cos 2y}{2}\,dy = \frac{\pi}{2}\left[y - \frac{\sin 2y}{2}\right]_0^{\pi/2} = \frac{\pi^2}{4}$

**6**. (10 points) A cone is made by revolving the line $y = 2x$ about the $y$-axis for $0 \le y \le 6$ cm. It is filled with water up to a depth of 4 cm. It is sucked out a straw which reaches 3 cm above the top of the cone. How much work is done? Give your answer as a multiple of $g\delta$ where $g$ is the acceleration of gravity and $\delta$ is the density.

**Solution**: The radius is $r = x = \frac{y}{2}$. The cross sectional area is $A = \pi r^2 = \pi \frac{y^2}{4}$.

The slice of water at height $y$ with thickness $dy$ has volume $dV = A\,dy = \frac{\pi}{4} y^2\,dy$ and weight $dF = g\delta dV = g\delta \frac{\pi}{4} y^2\,dy$. The spout is at height $y = 9$. So the water at height $y$ is lifted a distance $D = 9 - y$. There is water at heights $0 \le y \le 4$. So the work done is

$$W = \int D\,dF = \int_0^4 (9-y) g\delta \frac{\pi}{4} y^2\,dy = \frac{g\delta\pi}{4} \int_0^4 (9y^2 - y^3)\,dy = \frac{g\delta\pi}{4}\left[3y^3 - \frac{y^4}{4}\right]_0^4$$

$$= g\delta\pi(3 \cdot 4^2 - 4^2) = 32 g\delta\pi$$

**7**. (10 points) Solve the initial value problem:

$$\frac{dy}{dx} = \frac{x^2}{y^2} \qquad\qquad y(1) = 3$$

Find the general (explicit) solution and then find $y(0)$.

**Solution**: The equation separates:

$$\int y^2\,dy = \int x^2\,dx$$

$$\frac{y^3}{3} = \frac{x^3}{3} + C$$

We find $C$ by using the initial condition $x = 1$ when $y = 3$:

$$\frac{3^3}{3} = \frac{1^3}{3} + C$$

$$C = 9 - \frac{1}{3} = \frac{26}{3}$$

We substitute back and solve for $y$:

$$\frac{y^3}{3} = \frac{x^3}{3} + \frac{26}{3}$$

$$y = \sqrt[3]{x^3 + 26}$$

Finally, $y(0) = \sqrt[3]{26}$

**8**. (10 points) Solve the initial value problem:

$$\frac{dy}{dx} = 2xy + e^{x^2} \qquad\qquad y(0) = 4$$

Find the general (explicit) solution and then $y(1)$.

**Solution**: The equation is linear. Its standard form is $\frac{dy}{dx} - 2xy = e^{x^2}$. We find the integrating factor:

$$P = -2x \qquad \int P\,dx = \int -2x\,dx = -x^2 \qquad I = e^{-x^2}$$

We multiply the standard form by the integrating factor, check the left side is the derivative of a product and integrate:

$$e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y = e^{x^2}e^{-x^2}$$

$$\frac{d}{dx}\left(e^{-x^2}y\right) = 1$$

$$e^{-x^2}y = x + C$$

We find $C$ by using the initial condition $x = 0$ when $y = 4$:

$$e^{-0}4 = 0 + C$$

$$C = 4$$

We substitute back and solve for $y$:

$$e^{-x^2}y = x + 4$$

$$y = (x+4)e^{x^2}$$

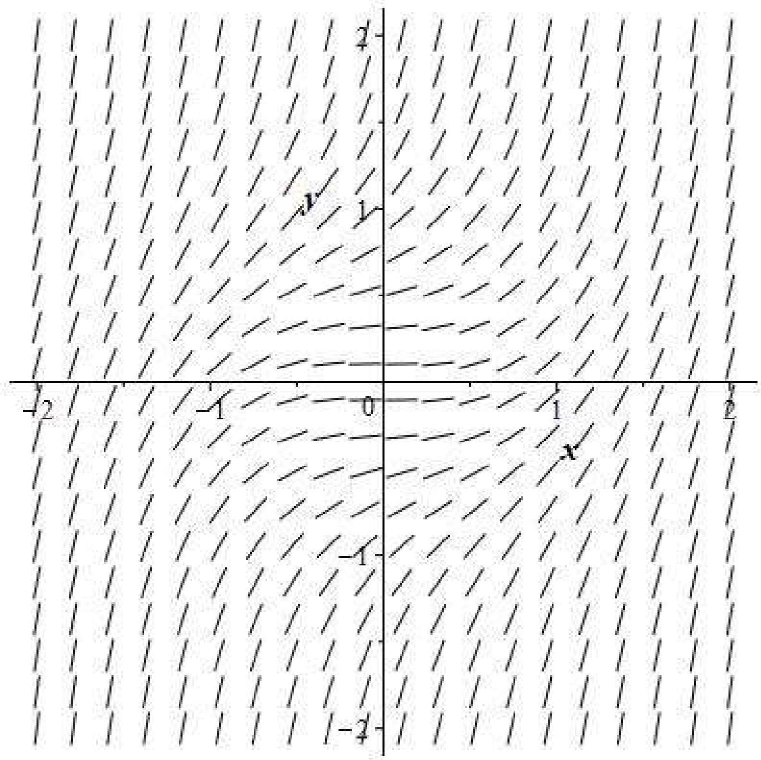
Finally, $y(1) = 5e$

**9**. (5 points) The plot at the right is the slope field for the differential equation

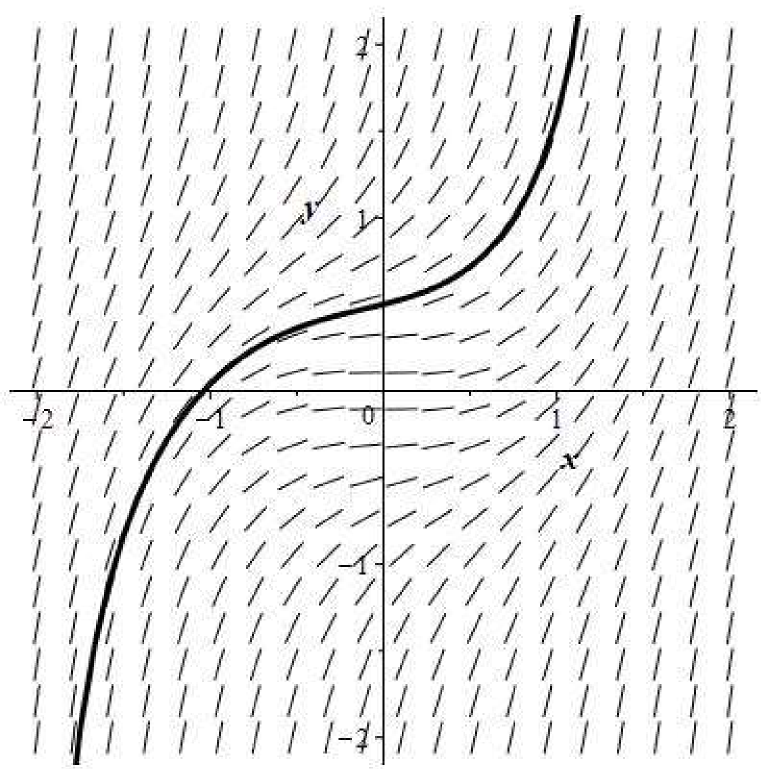
$$\frac{dy}{dx} = x^2 + y^2$$

On the plot, draw the solution curve satisfying the initial condition

$$y(0) = \frac{1}{2}$$




**Solution**: Start at $\left(0, \frac{1}{2}\right)$ and move so the curve is always tangent to the slope field.



**10**. (20 points) A pot contains $1000$ L of sugar water with a concentration of $0.01\ \frac{\text{kg sugar}}{\text{L water}}$. Sugar water with a concentration of $0.04\frac{\text{kg sugar}}{\text{L water}}$ is poured into the pot at $50\ \frac{\text{L}}{\text{min}}$. The sugar water is kept mixed and drains from the tank at $50\ \frac{\text{L}}{\text{min}}$.

Let $S(t)$ be the kg of sugar in the pot at time $t$.

**a**. How much sugar is in the tank at $t = 0$?

**Solution**: $S(0) = 1000\text{ L} \times 0.01\ \frac{\text{kg}}{\text{L}}$ $\boxed{S(0) = 10\text{ kg}}$

**b**. What is the differential equation for the rate of change of $S(t)$?

**Solution**: $\frac{dS}{dt} = \left[\text{rate Sugar in}\right] - \left[\text{rate Sugar out}\right] = 0.04\frac{\text{kg}}{\text{L}}50\frac{\text{L}}{\text{min}} - \frac{S(t)\text{ kg}}{1000\text{ L}}50\frac{\text{L}}{\text{min}}$

$$\boxed{\frac{dS}{dt} = 2 - 0.05\,S(t)}$$

**c**. How much sugar is in the pot at time $t$?

**Linear Solution**: Standard form is: $\frac{dS}{dt} + .05\,S = 2$

Integrating factor is: $P = .05 \quad I = e^{\int P dt} = e^{.05t}$

We multiply Standard form by Integrating factor: $e^{0.05t}\frac{dS}{dt} + .05e^{.05t}S = 2e^{.05t}$

We integrate: $e^{.05t}S = \frac{2}{.05}e^{.05t} + C = 40e^{.05t} + C$

We find $C$ using $S = 10$ when $t = 0$: $10 = 40 + C \quad C = -30$

We solve: $e^{.05t}S = 40e^{.05t} - 30$ $\boxed{S = 40 - 30e^{-.05t}}$

**Separable Solution**: We separate and integrate

$$\int \frac{dS}{2 - 0.05\,S} = \int dt \quad \Rightarrow \quad \frac{-1}{.05}\ln|2 - 0.05\,S| = t + C$$

$$\Rightarrow \quad \ln|2 - 0.05\,S| = -.05t - .05C \quad \Rightarrow \quad |2 - 0.05\,S| = e^{-.05C}e^{-.05t}$$

$$\Rightarrow \quad 2 - 0.05\,S = Ae^{-.05t}$$

We find $A$ using $S = 10$ when $t = 0$: $2 - 0.05 \times 10 = A = 1.5$

We solve: $2 - 0.05\,S = 1.5e^{-.05t} \qquad 0.05\,S = 2 - 1.5e^{-.05t}$ $\boxed{S = 40 - 30e^{-.05t}}$

**d**. Is the sugar in the pot increasing or decreasing with time?

**Solution**: $\frac{dS}{dt} = -30e^{-.05t}(-.05) = 1.5e^{-.05t} > 0$ $\boxed{\text{Increasing}}$