Name________________________________

MATH 172 Honors   Exam 3   Spring 2022

Sections 200   Solutions   P. Yasskin

1	/10	5	/20
2	/10	6	/10
3	/20	7	/10
4	/20	8	/10
		Total	/110

All Work Out

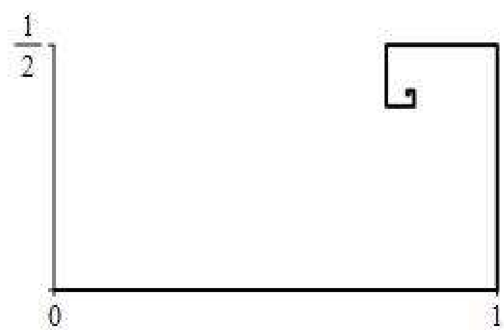
Points indicated. Part credit possible. Show all work.

1. (10 points) Compute $L = \lim\limits_{n\to\infty}\left(\dfrac{3}{n^4}\right)^{2/\ln n}$.

Solution: $\ln L = \lim\limits_{n\to\infty} \ln\left(\dfrac{3}{n^4}\right)^{2/\ln n} = \lim\limits_{n\to\infty} \dfrac{2}{\ln n}\ln\left(\dfrac{3}{n^4}\right) \overset{\text{l'H}}{=} \lim\limits_{n\to\infty} \dfrac{\dfrac{2}{\frac{3}{n^4}}\left(\dfrac{-12}{n^5}\right)}{\dfrac{1}{n}}$

$= \lim\limits_{n\to\infty} \dfrac{2n}{\frac{3}{n^4}}\left(\dfrac{-12}{n^5}\right) = \lim\limits_{n\to\infty}\dfrac{-24}{3} = -8 \qquad L = e^{-8}$

2. (10 points) This rectangular spiral is made by starting at $(0,0)$, moving right by 1, up by $\frac{1}{2}$, left by $\frac{1}{4}$, down by $\frac{1}{8}$, and repeating with each step being $\frac{1}{2}$ as long the previous step.
Find the coordinates of the limit point.



Solution: The x coordinate starts at 0 increases by 1, decreases by $\frac{1}{4}$, increases by $\frac{1}{16}$, etc. This is a geometric series with $a = 1$ and $r = \frac{-1}{4}$. So $x = \sum\limits_{n=0}^{\infty}\left(\frac{-1}{4}\right)^n = \dfrac{1}{1+\frac{1}{4}} = \frac{4}{5}$.

The y coordinate starts at 0 increases by $\frac{1}{2}$, decreases by $\frac{1}{8}$, increases by $\frac{1}{32}$, etc.

This is a geometric series with $a = \frac{1}{2}$ and $r = \frac{-1}{4}$. So $y = \sum\limits_{n=0}^{\infty}\frac{1}{2}\left(\frac{-1}{4}\right)^n = \dfrac{\frac{1}{2}}{1+\frac{1}{4}} = \frac{2}{5}$.

The limit point is $(x,y) = \left(\frac{4}{5},\frac{2}{5}\right)$.

3. (20 points) Determine whether each series is absolutely convergent, conditionally convergent or divergent. Be sure to name any convergence test(s) you use and check out all of its conditions:

a. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 + \ln n}{n^3 + \ln n}$

Solution: We compare to $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$ which diverges because it is the harmonic series.

We use the Limit Comparison Test.

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 + \ln n}{n^3 + \ln n}\,\frac{n}{1} = \lim_{n\to\infty} \frac{n^3 + n\ln n}{n^3 + \ln n} = \lim_{n\to\infty} \frac{1 + \dfrac{\ln n}{n^2}}{1 + \dfrac{\ln n}{n^3}} = 1$$

Since $0 < L = 1 < \infty$, the original series also diverges.

b. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{(n^2+3n)^2}$

Solution: We apply the Integral Test:

$$\int_1^\infty \frac{2n+3}{(n^2+3n)^2}\,dn = \left[\frac{-1}{n^2+3n}\right]_1^\infty = 0 - \frac{-1}{4} = \frac{1}{4}$$

Since the integral converges, the series also converges. Since the terms of the series are positive, it is also absolutely convergent.

c. $\displaystyle\sum_{n=2}^{\infty} (-1)^{n+1} \frac{n+1}{n-1}$

Solution: The Alternating Series Test fails because $\displaystyle\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n+1}{n-1} = 1 \neq 0.$

But this limit also says the series diverges by the n^{th}-Term Divergence test.

d. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2}{n^{3/4}}$

Solution: We apply the Alternating Series Test.

(1) alternating because $a_n = (-1)^n b_n$ with $b_n = \dfrac{2}{n^{3/4}} > 0$

(2) decreasing in absolute value because $b_n = \dfrac{2}{n^{3/4}}$ gets smaller as n gets larger.

(3) $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2}{n^{3/4}} = 0$ So the series converges.

The related absolute series is $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{3/4}}$ which is a p-series with $p = \frac{3}{4} < 1$ and so is divergent.

So the original series is conditionally convergent.

4. (20 points) Consider the sequence recursively defined by $a_{n+1} = 5 - \frac{4}{a_n}$ starting from $a_1 = 2$. Prove the limit exist and find it. (You may assume $a_n > 0$ without proof.)

a. Write out the first 3 terms:

$a_1 =$ $a_2 =$ $a_3 =$

Answer: $\boxed{a_1 = 2 \qquad a_2 = 3 \qquad a_3 = 5 - \frac{4}{3} = \frac{11}{3} = 3.67}$

b. Assuming the limit exists, find the possible values.

Solution: Let $L = \lim_{n\to\infty} a_n$. Then $\lim_{n\to\infty} a_{n+1} = L$ also. Take the limit of the recursion formula.

$L = 5 - \frac{4}{L} \quad \Rightarrow \quad L^2 = 5L - 4 \quad \Rightarrow \quad L^2 - 5L + 4 = 0 \quad \Rightarrow \quad (L-4)(L-1) = 0$

$L = 1$ or 4

c. What do you need to prove?

Circle one: increasing decreasing

Circle one and fill in the blank: bounded above by ______ bounded below by ______

Answer: increasing and bounded above by 4 or anything larger.

d. Prove it is bounded above or below:

Solution: We want to prove bounded above by 4 or $a_n \le 4$.

Initialization Step: $a_1 = 2 \le 4$

Induction Step: Assume $a_k \le 4$. We want to prove $a_{k+1} \le 4$.

Proof: $a_k \le 4 \quad \Rightarrow \quad \frac{4}{a_k} \ge \frac{4}{4} = 1 \quad \Rightarrow \quad -\frac{4}{a_k} \le -1$

$\Rightarrow \quad 5 - \frac{4}{a_k} \le 5 - 1 = 4 \quad \Rightarrow \quad a_{k+1} \le 4$

e. Prove it is increasing or decreasing:

Solution: We want to prove increasing or $a_{n+1} > a_n$.

Initialization Step: $a_2 = 3 > 2 = a_1$

Induction Step: Assume $a_{k+1} > a_k$. We want to prove $a_{k+2} > a_{k+1}$.

Proof: $a_{k+1} > a_k \quad \Rightarrow \quad \frac{4}{a_{k+1}} < \frac{4}{a_k} \quad \Rightarrow \quad -\frac{4}{a_{k+1}} > -\frac{4}{a_k}$

$\Rightarrow \quad 5 - \frac{4}{a_{k+1}} > 5 - \frac{4}{a_k} \quad \Rightarrow \quad a_{k+2} > a_{k+1}$

f. What do you conclude. What Theorem did you use?

Solution: By the Bounded Monotonic Sequence Theorem, the sequence converges.
Since it starts from 2 and increases and the only possible limits are 1 and 4, the limit must be 4.

5. (20 points) Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)2^n}(x-6)^n$.

a. Find the radius of convergence and state the open interval of absolute convergence.

$R =$ ____. Absolutely convergent on (_____,_____).

Solution: We use the Ratio Test. $|a_n| = \frac{n|x-6|^n}{(n^2+1)2^n} \quad |a_{n+1}| = \frac{(n+1)|x-6|^{n+1}}{\left((n+1)^2+1\right)2^{n+1}}$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)|x-6|^{n+1}}{\left((n+1)^2+1\right)2^{n+1}} \frac{(n^2+1)2^n}{n|x-6|^n} = \frac{|x-6|}{2} > 1$$

$|x-6| < 2$ So $R = 2$. Absolutely convergent on $(4, 8)$

b. Check the **Left** Endpoint:

$x =$_____ Write the series:_________________________ Circle one:

Reasons: $\boxed{\text{Convergent}}$

Divergent

Solution: $x = 4$: $$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)2^n}(-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n}{(n^2+1)}$$

This converges by the Alternating Series Test because $\frac{n}{(n^2+1)}$ is positive, decreasing and $\lim_{n\to\infty} \frac{n}{(n^2+1)} = 0$.

c. Check the **Right** Endpoint:

$x =$_____ Write the series:_________________________ Circle one:

Reasons: Convergent

$\boxed{\text{Divergent}}$

Solution: $x = 8$: $$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)2^n}(2)^n = \sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

We apply the Integral Test. $\int_1^{\infty} \frac{n}{n^2+1}\, dn = \left[\ln(n^2+1)\right]_1^{\infty} = \infty$

Since the integral diverges, the series diverges.

d. State the Interval of Convergence. Interval= ____________

Solution: The Interval of Convergence.is: $[4, 8)$

6. (10 points) Compute $\sum_{n=1}^{\infty}\left[\sec\left(\frac{1}{n}\right) - \sec\left(\frac{1}{n+1}\right)\right]$.

Solution: The series is telescoping. The partial sum is:

$$S_k = \sum_{n=1}^{k}\left[\sec\left(\frac{1}{n}\right) - \sec\left(\frac{1}{n+1}\right)\right]$$
$$= \left(\sec\left(\frac{1}{1}\right) - \sec\left(\frac{1}{2}\right)\right) + \left(\sec\left(\frac{1}{2}\right) - \sec\left(\frac{1}{3}\right)\right) + \cdots + \left(\sec\left(\frac{1}{k}\right) - \sec\left(\frac{1}{k+1}\right)\right)$$
$$= \sec(1) - \sec\left(\frac{1}{k+1}\right)$$
$$S = \lim_{n\to\infty} S_k = \lim_{n\to\infty}\left(\sec(1) - \sec\left(\frac{1}{k+1}\right)\right) = \sec(1) - \sec(0) = \sec(1) - 1$$

7. (10 points) Find the Maclaurin series for $f(x) = \frac{\sin(x^2)}{x}$.
Give the answer in both summation form and $\cdots$ form with at least 3 terms.
Then find $f^{(9)}(0)$, the 9^{th} derivative at 0.

Solution: We start from the Maclaurin series for $\sin u$, substitute $u = x^2$ and divide by x.

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!}\cdots = \sum_{k=1}^{\infty}\frac{(-1)^k u^{2k+1}}{(2k+1)!}$$
$$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}\cdots = \sum_{k=1}^{\infty}\frac{(-1)^k x^{4k+2}}{(2k+1)!}$$
$$f(x) = \frac{\sin(x^2)}{x} = x - \frac{x^5}{3!} + \frac{x^9}{5!} - \frac{x^{13}}{7!}\cdots = \sum_{k=1}^{\infty}\frac{(-1)^k x^{4k+1}}{(2k+1)!}$$

We compare this to the general Maclaurin series: $f(x) = \sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

So $n = 9 = 4k+1$ when $k = 2$. The $n = 9$ term is $\frac{f^{(9)}(0)}{9!}x^9 = \frac{(-1)^2 x^{4\cdot 2+1}}{(2\cdot 2+1)!} = \frac{x^9}{5!}$.

So $f^{(9)}(0) = \frac{9!}{5!}$.

8. (10 points) Compute $\lim_{x\to 0}\frac{\cos(x^2) - 1 + \frac{x^4}{2}}{x^8}$

Solution: $\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \cdots \qquad \cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots$

$$\lim_{x\to 0}\frac{\cos(x^2) - 1 + \frac{x^4}{2}}{x^8} = \lim_{x\to 0}\frac{\left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \cdots\right) - 1 + \frac{x^4}{2}}{x^8} = \frac{1}{4!}$$