Name________________________________

MATH 172 Honors Final Exam Spring 2022

Sections 200 Solutions P. Yasskin

1-10	/50	14	/10
11	/10	15	/10
12	/5	16	/15
13	/5	Total	/105

Multiple Choice: (5 points each. Circle your answers.)
Show your work in case I want to look at it.

1. (5 points) Find the average value of the function $f(x) = x\sqrt{x^2+1}$ on the interval $\left[0, \sqrt{8}\,\right]$.

a. $\dfrac{13}{3\sqrt{8}}$ d. $\boxed{\dfrac{26}{3\sqrt{8}}}$ g. $\dfrac{13}{\sqrt{8}}$ j. $\dfrac{52}{3\sqrt{8}}$

b. $\dfrac{13}{3}$ e. $\dfrac{26}{3}$ h. 13 k. $\dfrac{52}{3}$

c. $\dfrac{13\sqrt{8}}{3}$ f. $\dfrac{26\sqrt{8}}{3}$ i. $13\sqrt{8}$ l. $\dfrac{52\sqrt{8}}{3}$

Solution: $f_{\text{ave}} = \dfrac{1}{\sqrt{8}}\displaystyle\int_0^{\sqrt{8}} x\sqrt{x^2+1}\,dx = \dfrac{1}{\sqrt{8}}\left[\dfrac{1}{3}(x^2+1)^{3/2}\right]_0^{\sqrt{8}} = \dfrac{1}{\sqrt{8}}\left(\dfrac{1}{3}(9)^{3/2} - \dfrac{1}{3}\right) = \dfrac{26}{3\sqrt{8}}$

2. (5 points) Find the arclength of the curve $\vec{r}(t) = \langle t\cos t - \sin t, t\sin t + \cos t\rangle$ between $t = 0$ and $t = 1$.

a. $\frac{1}{2}\ln\left(\sqrt{2}+1\right) - \frac{1}{2}\sqrt{2}$ d. $\ln\left(\sqrt{2}+1\right) - \sqrt{2}$ g. $\dfrac{1}{8}$ j. $\dfrac{\sqrt{2}}{8}$

b. $\frac{1}{2}\ln\left(\sqrt{2}+1\right) + \frac{1}{2}\sqrt{2}$ e. $\ln\left(\sqrt{2}+1\right) + \sqrt{2}$ h. $\dfrac{1}{4}$ k. $\dfrac{\sqrt{2}}{4}$

c. $\frac{1}{2}\ln\left(\sqrt{2}+1\right)$ f. $\ln\left(\sqrt{2}+1\right)$ i. $\boxed{\dfrac{1}{2}}$ l. $\dfrac{\sqrt{2}}{2}$

Solution: $\dfrac{dx}{dt} = \cos t - t\sin t - \cos t = -t\sin t$ $\dfrac{dy}{dt} = \sin t + t\cos t - \sin t = t\cos t$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{(-t\sin t)^2 + (t\cos t)^2}\,dt = \int_0^1 t\sqrt{\sin^2 t + \cos^2 t}\,dt = \int_0^1 t\,dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

3. (5 points) A 60 lb force stretches a spring 3 ft from its rest position.
How much work is done to stretch the spring from its rest position to 4 ft from its rest position?

a. 12 ft-lb	d. 60 ft-lb	g. **[160 ft-lb]**	j. 320 ft-lb
b. 20 ft-lb	e. 80 ft-lb	h. 180 ft-lb	k. 720 ft-lb
c. 24 ft-lb	f. 90 ft-lb	i. 240 ft-lb	l. 1440 ft-lb

Solution: The force is $F = kx$ or $60 = k3$. So $k = 20$ and so $F = 20x$.
Then the work is

$$W = \int F\,dx = \int_0^4 20x\,dx = \left[10x^2\right]_0^4 = 160$$

4. (5 points) The base of a solid is the region between $y = 4 - x^2$ and the x-axis.
The cross sections perpendicular to the x-axis are squares. Find its volume.

a. $\frac{32}{35}$	e. $\frac{32}{15}$	i. $\frac{32}{5}$	m. $\frac{32}{3}$
b. $\frac{128}{35}$	f. $\frac{128}{15}$	j. $\frac{128}{5}$	n. $\frac{128}{3}$
c. $\frac{256}{35}$	g. $\frac{256}{15}$	k. $\frac{256}{5}$	o. $\frac{256}{3}$
d. $\frac{512}{35}$	h. $\boxed{\frac{512}{15}}$	l. $\frac{512}{5}$	p. $\frac{512}{3}$

Solution: The parabola $y = 4 - x^2$ intersects the x-axis at $x = -2$ and 2.
The side of a square is $s = y = 4 - x^2$. So its area is $A = s^2 = (4 - x^2)^2 = 16 - 8x^2 + x^4$.
So the volume is

$$V = \int_{-2}^{2} A\,dx = \int_{-2}^{2} (16 - 8x^2 + x^4)\,dx = \left[16x - 8\frac{x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2} = 2\left(32 - \frac{64}{3} + \frac{32}{5}\right)$$

$$= 64\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{512}{15}$$

5. (5 points) Compute $\int_0^2 2x^3 e^{x^2} dx$.

a. $\frac{3}{2}e^4 - 1$ d. $3e^4 - 1$ g. $4e^4 - 1$ j. $5e^4 - 1$

b. $\frac{3}{2}e^4$ e. $3e^4$ h. $4e^4$ k. $5e^4$

c. $\frac{3}{2}e^4 + 1$ f. $\boxed{3e^4 + 1}$ i. $4e^4 + 1$ l. $5e^4 + 1$

Solution: First make the substitution $w = x^2$ and $dw = 2x\,dx$.

$$\int 2x^3 e^{x^2} dx = \int w e^w dw$$

We now do integration by parts with $\begin{array}{ll} u = w & dv = e^w dw \\ du = dw & v = e^w \end{array}$.

$$\int 2x^3 e^{x^2} dx = we^w - \int e^w dw = we^w - e^w + C = x^2 e^{x^2} - e^{x^2} + C$$

$$\int_0^2 2x^3 e^{x^2} dx = \left[x^2 e^{x^2} - e^{x^2}\right]_0^2 = (4e^4 - e^4) - (0 - e^0) = 3e^4 + 1$$

6. (5 points) After making the appropriate trig substitution, $\int_0^1 \frac{1}{(x^2-4)^8} dx$ becomes

a. $\frac{1}{2^{15}} \int_{x=0}^{1} \sec^9\theta\, d\theta$ e. $\boxed{\frac{1}{2^{15}} \int_{x=0}^{1} \sec^{15}\theta\, d\theta}$ i. $\frac{1}{2^{15}} \int_{x=0}^{1} \sec^{15}\theta \tan\theta\, d\theta$

b. $\frac{1}{2^{15}} \int_{x=0}^{1} \frac{1}{\sec^{14}\theta} d\theta$ f. $\frac{1}{2^{15}} \int_{x=0}^{1} \frac{\sec\theta}{\tan^{15}\theta} d\theta$ j. $\frac{1}{2^{15}} \int_{x=0}^{1} \frac{\tan\theta}{\sec^{14}\theta} d\theta$

c. $\frac{1}{2^{16}} \int_{x=0}^{1} \sec^{10}\theta\, d\theta$ g. $\frac{1}{2^{16}} \int_{x=0}^{1} \sec^{16}\theta\, d\theta$ k. $\frac{1}{2^{16}} \int_{x=0}^{1} \sec^{16}\theta \tan\theta\, d\theta$

d. $\frac{1}{2^{16}} \int_{x=0}^{1} \frac{1}{\sec^{16}\theta} d\theta$ h. $\frac{1}{2^{16}} \int_{x=0}^{1} \frac{1}{\tan^{16}\theta} d\theta$ l. $\frac{1}{2^{16}} \int_{x=0}^{1} \frac{\tan\theta}{\sec^{16}\theta} d\theta$

Solution: The minus says this is a $\sin$ or $\sec$ trig substitution.

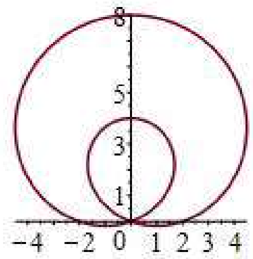
Since $0 \le x \le 1$, it cannot be a $\sec$ substitution.

So $x = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$.

$$\int_0^1 \frac{1}{(x^2-4)^8} dx = \int_{x=0}^{1} \frac{1}{(4\sin^2\theta - 4)^8} 2\cos\theta\, d\theta = \frac{1}{2^{15}} \int_{x=0}^{1} \frac{1}{(\cos^2\theta)^8} \cos\theta\, d\theta = \frac{1}{2^{15}} \int_{x=0}^{1} \sec^{15}\theta\, d\theta$$

7. (5 points) Which of the following is the graph of the polar equation $r = 2 - 6\sin\theta$?

a.



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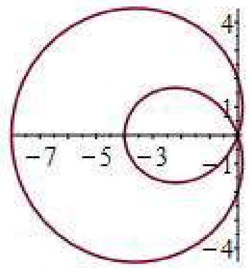
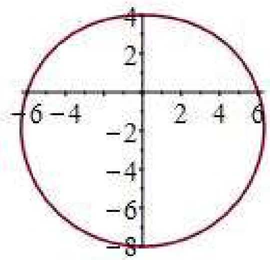
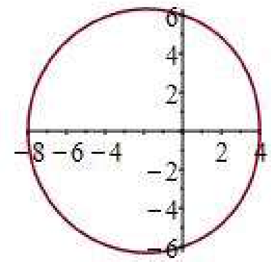


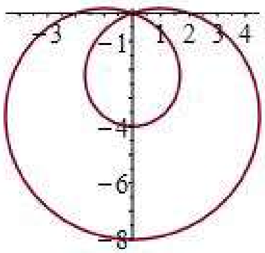
e.



g.



b.



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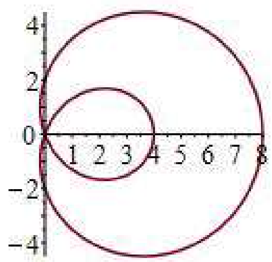

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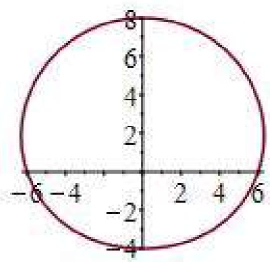
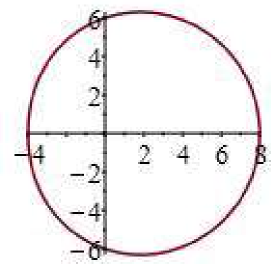


h.



Solution: $r(0) = 2 \qquad r\left(\frac{\pi}{2}\right) = -4 \qquad r(\pi) = 2 \qquad r\left(\frac{3\pi}{2}\right) = 8$ That's plot (b).

8. (5 points) Solve the initial value problem $\frac{dy}{dx} = \frac{y^{2/3}}{x^{2/3}}$ with $y(0) = 1$. Then find $y(1)$.

a. $y(1) = 0$ d. $y(1) = 2$ g. $y(1) = 2^{1/3}$ j. $y(1) = 2^{1/3} + 1$

b. $y(1) = 1$ e. $y(1) = 4$ h. $y(1) = 4^{1/3}$ k. $y(1) = 4^{1/3} + 1$

c. $y(1) = e$ f. $y(1) = 8$ i. $y(1) = e^{1/3}$ l. $y(1) = e^{1/3} + 1$

Solution: We separate the variables and integrate:

$$\int \frac{1}{y^{2/3}}\,dy = \int \frac{1}{x^{2/3}}\,dx \qquad 3y^{1/3} = 3x^{1/3} + C$$

We use the initial condition: $x = 0$ when $y = 1$:

$$3 = C \qquad 3y^{1/3} = 3x^{1/3} + 3 \qquad y = (x^{1/3} + 1)^3$$

Then $y(1) = (1+1)^3 = 8$

9. (5 points) Find the sum of the series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k}}{(2k)!}$.

a. $-\pi$ d. $\boxed{-1}$ g. 2

b. $-\frac{\pi}{2}$ e. 0 h. $\frac{\pi}{2}$

c. -2 f. 1 i. π

Solution: $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^n x^{2k}}{(2k)!} = \cos x$ So $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^n \pi^{2k}}{(2k)!} = \cos \pi = -1$

10. (5 points) If $f(x) = x^2 e^x$, find $f^{(11)}(0)$, the 11^{th} derivative at 0.

a. $9!$ d. $\frac{1}{9!}$ g. 72 j. $\frac{1}{72}$

b. $10!$ e. $\frac{1}{10!}$ h. 90 k. $\frac{1}{90}$

c. $11!$ f. $\frac{1}{11!}$ i. $\boxed{110}$ l. $\frac{1}{110}$

Solution: We find the Maclaurin series for $f(x)$. $\displaystyle e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ $\displaystyle f(x) = x^2 e^x = \sum_{k=0}^{\infty} \frac{x^{k+2}}{k!}$

The general Maclaurin is $\displaystyle f(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

The derivative $f^{(11)}(0)$ occurs in the term with $n = 11$ and $k = 9$.

We equate them: $\displaystyle\frac{f^{(11)}(0)}{11!} x^{11} = \frac{x^{11}}{9!}$ So $\displaystyle f^{(11)}(0) = \frac{11!}{9!} = 11 \cdot 10 = 110$.

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (10 points) Find the partial fraction expansion for $\dfrac{3x-2}{x^3+x}$ and compute $\displaystyle\int \frac{3x-2}{x^3+x}\,dx$.

Solution: The general partial fraction expansion is: $\dfrac{3x-2}{x^3+x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

We clear the denominator and expand: $3x-2 = A(x^2+1)+(Bx+C)x = (A+B)x^2+Cx+A$

We equate the coefficients of each power of x: $A+B=0$ $C=3$ $A=-2$ So $B=2$.

So the partial fraction expansion is: $\dfrac{3x-2}{x^3+x} = \dfrac{-2}{x} + \dfrac{2x+3}{x^2+1}$ So:

$$\int \frac{3x-2}{x^3+x}\,dx = \int \frac{-2}{x} + \frac{2x}{x^2+1} + \frac{3}{x^2+1}\,dx = -2\ln|x| + \ln(x^2+1) + 3\arctan x + C$$

12. (5 points) Solve the initial value problem $\dfrac{dy}{dx} + \dfrac{y}{x} + x^3 = 0$ with $y(5)=0$. Then find $y(-5)$.

Solution: Standard linear form: $\dfrac{dy}{dx} + \dfrac{y}{x} = -x^3$ $P = \dfrac{1}{x}$ $I = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

Multiply the standard form by x: $x\dfrac{dy}{dx} + y = -x^4$

Recognize the left side as the derivative of a product and integrate:

$$\frac{d}{dx}(xy) = -x^4 \qquad xy = -\int x^4\,dx = -\frac{x^5}{5} + C$$

Use the initial condition: $x=5$ when $y=0$:

$$0 = -\frac{5^5}{5} + C \qquad C = 5^4 \qquad xy = -\frac{x^5}{5} + 5^4 \qquad y = -\frac{x^4}{5} + \frac{5^4}{x}$$

$$y(-5) = -\frac{(-5)^4}{5} + \frac{5^4}{-5} = -5^3 - 5^3 = -2\cdot 5^3 = -250$$

13. (5 points) The series $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$ converges to the function $\dfrac{1}{(1-x)^2}$ on $(-1,1)$.

What function does the series $\displaystyle\sum_{n=0}^{\infty}(n+1)nx^{n-1}$ converges to on $(-1,1)$?

Solution: Since $\displaystyle\sum_{n=0}^{\infty}(n+1)nx^{n-1}$ is the derivative of $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$, it converges to

$$\frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{2}{(1-x)^3}.$$

14. (10 points) Compute $\lim_{x \to 0} \frac{\sin x - x\cos x}{x^3}$.

Solution: $\sin x - x\cos x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$

$= \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + (*)x^5 + \cdots = \frac{1}{3}x^3 + (*)x^5 + \cdots$

$$\lim_{x \to 0} \frac{\sin x - x\cos x}{x^3} = \lim_{x \to 0} \frac{\frac{1}{3}x^3 + (*)x^5 + \cdots}{x^3} = \frac{1}{3}$$

15. (10 points) Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{(x-4)^n}{3^n(2n+1)^3}$.

Be sure to explain any test you use.

Solution: Ratio test: $|a_n| = \frac{|x-4|^n}{3^n(2n+1)^3}$ $\qquad |a_{n+1}| = \frac{|x-4|^{n+1}}{3^{n+1}(2n+3)^3}$

$$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{|x-4|^{n+1}}{3^{n+1}(2n+3)^3} \frac{3^n(2n+1)^3}{|x-4|^n} = \frac{|x-4|}{3} \lim_{n \to \infty} \left(\frac{2n+1}{2n+3}\right)^3 = \frac{|x-4|}{3} < 1$$

So the series converges absolutely on $|x-4| < 3$ or $(1, 7)$. We check endpoints.

$x = 1$: $\sum_{n=0}^{\infty} \frac{(1-4)^n}{3^n(2n+1)^3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$ Converges by the Alternating Series Test.

$x = 7$: $\sum_{n=0}^{\infty} \frac{(7-4)^n}{3^n(2n+1)^3} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}$ Compare to $\sum_{n=0}^{\infty} \frac{1}{8n^3}$ which

is a convergent p-series with $p = 3 > 1$

Since $\frac{1}{(2n+1)^3} < \frac{1}{8n^3}$, the series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}$ converges by the Simple Comparison Test.

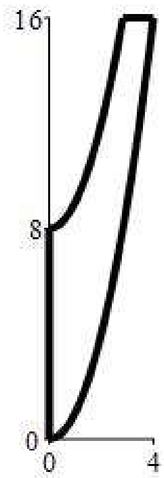
So the interval of convergence is $[1, 7]$.

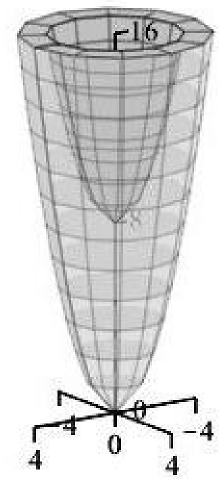
16. (15 points) The region in the first quadrant bounded by

$x = \sqrt{y}$, $x = \sqrt{y-8}$, $x = 0$ and $y = 16$

is revolved about the y-axis to form a bowl.

HINT: You will need to split the integrals at $y = 8$.





a.(5 pts) Find the voume swept out.

Solution: We do a y-integral and split the integral at 8. The slices are horizontal and rotate into disks below 8 and washers above 8. The outer radius is $R = x = \sqrt{y}$. The inner radius is $r = x = \sqrt{y-8}$.

$$V = \int_0^8 \pi R^2\,dy + \int_8^{16} \pi(R^2 - r^2)\,dy = \int_0^8 \pi\left[\sqrt{y}\,\right]^2 dy + \int_8^{16} \pi\left(\left[\sqrt{y}\,\right]^2 - \left[\sqrt{y-8}\,\right]^2\right) dy$$

$$= \int_0^8 \pi y\,dy + \int_8^{16} \pi(y - y + 8)\,dy = \pi\left[\frac{y^2}{2}\right]_0^8 + 8\pi\left[y\right]_8^{16} = \pi 32 + \pi 64 = 96\pi$$

b. (5 pts) If the bowl is fillled with water, find the work done to pump the water out the top of the bowl. Give the answer as a multiple of $g\delta$ where g is the acceleration of gravity and δ is the density of water. Once you plug in numbers, you do not need to simplify.

Solution: The slice of water at height y is a disk of radius $r = x = \sqrt{y-8}$.
So its volume is $dV = \pi r^2\,dy = \pi(y-8)\,dy$ and its weight is $dF = g\delta\,dV = g\delta\pi(y-8)\,dy$.
The slice needs to be lifted $D = 16 - y$. So the work is

$$W = \int D\,dF = \int_8^{16} (16-y)g\delta\pi(y-8)\,dy = g\delta\pi\int_8^{16}(16y - 128 - y^2 + 8y)dy$$

$$= g\delta\pi\left[12y^2 - 128y - \frac{y^3}{3}\right]_8^{16}$$

$$= g\delta\pi\left(12\cdot 16^2 - 128\cdot 16 - \frac{16^3}{3}\right) - g\delta\pi\left(12\cdot 8^2 - 128\cdot 8 - \frac{8^3}{3}\right)$$

$$= g\delta\pi\left(3\cdot 2^{10} - 2^{11} - \frac{2^{12}}{3} - 3\cdot 2^8 + 2^{10} + \frac{2^9}{3}\right)$$

$$= g\delta\pi 2^8\left(12 - 8 - \frac{16}{3} - 3 + 4 + \frac{2}{3}\right) = \frac{256}{3}\pi g\delta$$

c. (5 pts Extra Credit) Find the height of the centroid of the bowl (without water). (The volume was found in part a.)

Solution: The 1^{st}-moment is the same as the volume integral but with an extra factor of y in the integrand.

$$V_1 = \int_0^8 \pi y R^2\,dy + \int_8^{16} \pi y(R^2 - r^2)\,dy = \int_0^8 \pi y^2\,dy + \int_8^{16} \pi 8y\,dy = \pi\left[\frac{y^3}{3}\right]_0^8 + \pi\left[4y^2\right]_8^{16}$$

$$= \pi\frac{8^3}{3} + \pi 4\cdot 16^2 - \pi 4\cdot 8^2 = \frac{2816}{3}\pi$$

$$\bar{y} = \frac{V_1}{V} = \frac{\pi\frac{8^3}{3} + \pi 4\cdot 16^2 - \pi 4\cdot 8^2}{96\pi} = \frac{2816\pi}{3}\frac{1}{96\pi} = \frac{88}{9}$$