Name________________________

1-13	/65	15	/10
14	/15	16	/15
		Total	/105

MATH 221 Exam 2 Fall 2009

Section 503 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Compute $\int_0^2 \int_{-x}^{x} 3y^2\,dy\,dx$.

a. 8 Correct Choice

b. 12

c. 16

d. 24

e. 0

$$\int_0^2 \int_{-x}^{x} 3y^2\,dy\,dx = \int_0^2 \left[y^3\right]_{y=-x}^{x} dx = \int_0^2 \left(x^3 - (-x)^3\right) dx = \int_0^2 2x^3\,dx = \left[\frac{x^4}{2}\right]_{x=0}^{2} = 8$$

2. Compute $\int_0^9 \int_{\sqrt{y}}^{3} \pi \sin(\pi x^3)\,dx\,dy$. HINT: Reverse the order of integration.

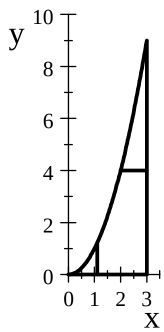
a. $\frac{1}{9}$

b. $\frac{2\pi}{9}$

c. $\frac{1}{3}$

d. $\frac{2}{3}$ Correct Choice

e. $\frac{2\pi}{3}$

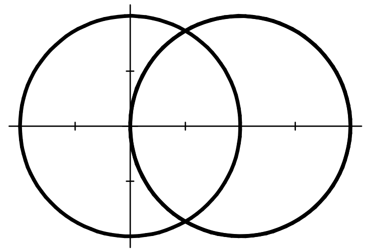


$$\int_0^9 \int_{\sqrt{y}}^{3} \pi \sin(\pi x^3)\,dx\,dy = \int_0^3 \int_0^{x^2} \pi \sin(\pi x^3)\,dy\,dx$$

$$= \int_0^3 [\pi \sin(\pi x^3) y]_{y=0}^{x^2} dx = \int_0^3 \pi x^2 \sin(\pi x^3)\,dx$$

$$= \frac{-1}{3} \cos(\pi x^3)\Big|_0^3 = \frac{-1}{3}\cos(27\pi) + \frac{1}{3}\cos(0) = \frac{2}{3}$$

3. Find the area inside the circle $r = 2\cos\theta$ but outside the circle $r = 1$.



a. $\frac{\pi}{3} - \cos\frac{\pi}{3}$

b. $\frac{2\pi}{3} + \cos\frac{2\pi}{3}$

c. $\frac{\pi}{3} + \sin\frac{2\pi}{3}$ Correct Choice

d. $\frac{2\pi}{3} - \sin\frac{\pi}{3}$

e. $\frac{2\pi}{3} + \sin\frac{\pi}{3}$

$$2\cos\theta = 1 \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = -\frac{\pi}{3}, \frac{\pi}{3}$$

$$A = \iint 1\,dA = \int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} r\,dr\,d\theta = \int_{-\pi/3}^{\pi/3}\left[\frac{r^2}{2}\right]_{r=1}^{2\cos\theta} d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(4\cos^2\theta - 1)\,d\theta$$
$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left(4\frac{1+\cos 2\theta}{2} - 1\right)d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3}(1 + 2\cos 2\theta)\,d\theta$$
$$= \frac{1}{2}\left[\theta + \sin 2\theta\right]_{\theta=-\pi/3}^{\pi/3} = \frac{1}{2}\left(\frac{\pi}{3} + \sin\frac{2\pi}{3}\right) - \frac{1}{2}\left(-\frac{\pi}{3} + \sin\frac{-2\pi}{3}\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{3} + \sin\frac{2\pi}{3} + \frac{\pi}{3} + \sin\frac{2\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3}$$

4. Compute $\int_0^2\int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx$. HINT: Switch to polar coordinates.

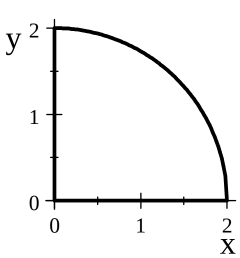
a. $\frac{\pi}{2}e^2$

b. $\frac{\pi}{2}e^4$

c. $\frac{\pi}{2}(e^2 - 1)$

d. $\frac{\pi}{2}(e^4 - 1)$

e. $\frac{\pi}{4}(e^4 - 1)$ Correct Choice



$$\int_0^2\int_0^{\sqrt{4-x^2}} e^{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^2 e^{r^2}\,r\,dr\,d\theta$$
$$= \left[\frac{\pi}{2}\right]\left[\frac{1}{2}e^{r^2}\right]_0^2 = \frac{\pi}{4}(e^4 - 1)$$

5. Compute the mass of the solid cone $\sqrt{x^2+y^2} \le z \le 4$ if the volume density is $\rho = z$.

a. 4π

b. 8π

c. 16π

d. 32π

e. 64π Correct Choice

$$M = \iiint \rho\, dV = \int_0^{2\pi}\int_0^4\int_r^4 z\, r\, dz\, dr\, d\theta = 2\pi \int_0^4 r\left[\frac{z^2}{2}\right]_{z=r}^4 dr = 2\pi \int_0^4 r\left(8 - \frac{r^2}{2}\right) dr$$
$$= 2\pi\left[4r^2 - \frac{r^4}{8}\right]_{r=0}^4 = 2\pi\left(4^3 - \frac{4^4}{8}\right) = 2\pi 4^3\left(1 - \frac{1}{2}\right) = 64\pi$$

6. Compute the center of mass of the solid cone $\sqrt{x^2+y^2} \le z \le 4$ if the volume density is $\rho = z$.

a. $\left(0, 0, \frac{5}{16}\right)$

b. $\left(0, 0, \frac{16}{5}\right)$ Correct Choice

c. $\left(0, 0, \frac{1024\pi}{5}\right)$

d. $\left(0, 0, \frac{5}{1024\pi}\right)$

e. $\left(0, 0, \frac{8}{5}\right)$

$$M_{xy} = \iiint z\rho\, dV = \int_0^{2\pi}\int_0^4\int_r^4 z^2\, r\, dz\, dr\, d\theta = 2\pi \int_0^4 r\left[\frac{z^3}{3}\right]_{z=r}^4 dr = 2\pi \int_0^4 r\left(\frac{64}{3} - \frac{r^3}{3}\right) dr$$
$$= \frac{2\pi}{3}\left[32r^2 - \frac{r^5}{5}\right]_{r=0}^4 = \frac{2\pi}{3}\left(2 \cdot 4^4 - \frac{4^5}{5}\right) = \frac{2\pi}{3}4^4\left(2 - \frac{4}{5}\right) = \frac{1024\pi}{5}$$

$\bar{z} = \dfrac{M_{xy}}{M} = \dfrac{1024\pi}{5}\dfrac{1}{64\pi} = \dfrac{16}{5}$ By symmetry $\bar{x} = \bar{y} = 0$

7. Find the average value of the function $f = \dfrac{1}{x^2+y^2+z^2}$ over the solid region between the two spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$. HINT: $f_{\text{ave}} = \dfrac{\iiint f\, dV}{\iiint 1\, dV}$

a. $\frac{3}{4}$

b. $\frac{5}{8}$

c. $\frac{3}{7}$ Correct Choice

d. 4π

e. 8π

$$\iiint f\, dV = \int_0^{\pi}\int_0^{2\pi}\int_1^2 \frac{1}{\rho^2}\rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi = \left[-\cos\varphi\right]_0^{\pi}(2\pi)(2-1) = 4\pi$$

$\iiint 1\, dV =$ Volume $= \dfrac{4}{3}\pi 2^3 - \dfrac{4}{3}\pi 1^3 = \dfrac{28}{3}\pi$ $\quad f_{\text{ave}} = \dfrac{4\pi 3}{28\pi} = \dfrac{3}{7}$

8. Compute $\iint \cos\left(\frac{x^2}{9}+\frac{y^2}{4}\right)dx\,dy$ over the region inside the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$.

HINT: Elliptical coordinates are $x=3t\cos\theta$, $y=2t\sin\theta$.

- **a.** $6\pi\sin(1)-6\pi$
- **b.** $6\pi\sin(1)$ Correct Choice
- **c.** $6\pi-6\pi\cos(1)$
- **d.** $-6\pi\cos(1)$
- **e.** $6\pi\cos(1)$

$$J=\left|\frac{\partial(x,y)}{\partial(t,\theta)}\right|=\left\|\begin{matrix}3\cos\theta & 2\sin\theta\\ -3t\sin\theta & 2t\cos\theta\end{matrix}\right\|=|6t\cos^2\theta+6t\sin^2\theta|=6t \text{ assuming } t\ge 0$$

$$\frac{x^2}{9}+\frac{y^2}{4}=\frac{(3t\cos\theta)^2}{9}+\frac{(2t\sin\theta)^2}{4}=t^2<1 \qquad 0\le t\le 1$$

$$\iint \cos\left(\frac{x^2}{9}+\frac{y^2}{4}\right)dx\,dy=\int_0^{2\pi}\int_0^1\cos(t^2)\,6t\,dt\,d\theta=2\pi 3\sin(t^2)\Big|_0^1=6\pi\sin(1)$$

9. Compute $\int_{(0,0)}^{(\sqrt{2},2)} x\,ds$ along the parabola $y=x^2$ parametrized as $\vec{r}(t)=(t,t^2)$.

- **a.** $\frac{13}{6}$ Correct Choice
- **b.** $\frac{9}{4}$
- **c.** $\frac{27}{4}$
- **d.** $\frac{1}{12}(17^{3/2}-1)$
- **e.** $\frac{17^{3/2}}{12}$

$$\vec{v}=(1,2t) \qquad |\vec{v}|=\sqrt{1+4t^2} \qquad x=t$$

$$\int_{(0,0)}^{(\sqrt{2},2)} x\,ds=\int_0^{\sqrt{2}} x|\vec{v}|\,dt=\int_0^{\sqrt{2}} t\sqrt{1+4t^2}\,dt=\left[\frac{(1+4t^2)^{3/2}}{12}\right]_0^{\sqrt{2}}=\frac{13}{6}$$

10. Compute $\int_{(0,0)}^{(\sqrt{2},2)} \vec{\nabla}f\cdot d\vec{s}$ for $f=xy$ along the parabola $y=x^2$ parametrized as $\vec{r}(t)=(t,t^2)$.

- **a.** 2
- **b.** $2\sqrt{2}$ Correct Choice
- **c.** 4
- **d.** $4\sqrt{2}$
- **e.** 8

$$\vec{v}=(1,2t) \qquad \vec{\nabla}f=(y,x)=(t^2,t) \qquad \vec{\nabla}f\cdot\vec{v}=t^2+2t^2=3t^2$$

$$\int_{(0,0)}^{(\sqrt{2},2)} \vec{\nabla}f\cdot d\vec{s}=\int_0^{\sqrt{2}}\vec{\nabla}f\cdot\vec{v}\,dt=\int_0^{\sqrt{2}}3t^2\,dt=\left[t^3\right]_0^{\sqrt{2}}=2\sqrt{2}$$

11. Compute $\int \vec{F} \cdot d\vec{s}$ for $\vec{F} = (-yz^2, xz^2, z^3)$ counterclockwise around the circle $x^2 + y^2 = 4$ with $z = 4$. HINT: Parametrize the circle.

a. 64

b. 128

c. 64π

d. 128π Correct Choice

e. 256π

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 4)$ $\vec{v} = (-2\sin\theta, 2\cos\theta, 0)$ $\vec{F}(\vec{r}(\theta)) = (-32\sin\theta, 32\cos\theta, 64)$

$\vec{F}(\vec{r}(\theta)) \cdot \vec{v} = 64\sin^2\theta + 64\cos^2\theta = 64$ $\int \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 64\, d\theta = 128\pi$

12. Compute $\iiint \vec{\nabla} \cdot \vec{F}\, dV$ for $\vec{F} = (xy^2, yz^2, zx^2)$ over the solid hemisphere $x^2 + y^2 + z^2 \le 25$ with $z \ge 0$.

a. $\frac{125}{3}\pi$

b. $\frac{250}{3}\pi$

c. $\frac{500}{3}\pi$

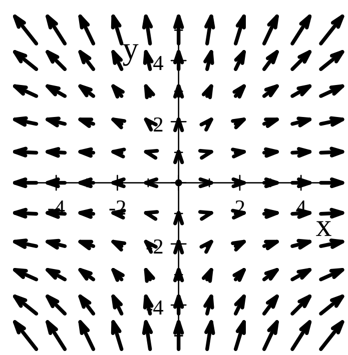
d. 1250π Correct Choice

e. 2500π

$\vec{\nabla} \cdot \vec{F} = y^2 + z^2 + x^2 = \rho^2$

$\iiint \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{\pi/2} \int_0^{2\pi} \int_0^5 \rho^2 \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi = \left[-\cos\varphi\right]_0^{\pi/2} (2\pi) \left[\frac{\rho^5}{5}\right]_0^5 = 2\pi 5^4 = 1250\pi$

13. Which of the following vector fields are plotted at the right?



a. $(y^2, 4x)$

b. $(-y^2, 4x)$

c. $(4x, y^2)$ Correct Choice

d. $(4x, -y^2)$

e. $(4xz, 4y)$

All of the vectors in the plot have a positive y-component. This only happens for vector field (c).

Work Out: (Points indicated. Part credit possible. Show all work.)

14. (15 points) Find the volume of the largest box such the length plus twice the width plus three times the height is 36.

Maximize $V = LWH$ subject to the constraint $g = L + 2W + 3H = 36$.

Lagrange Multiplier Method

$\vec{\nabla}V = (WH, LH, LW)$ $\quad \vec{\nabla}g = (1,2,3)$

The Lagrange equations are $\vec{\nabla}V = \lambda\vec{\nabla}g$ or

$$\begin{aligned} WH &= \lambda \\ LH &= 2\lambda \\ LW &= 3\lambda \end{aligned} \quad\Rightarrow\quad \begin{aligned} LH &= 2WH \\ LW &= 3WH \end{aligned} \quad\Rightarrow\quad \begin{aligned} W &= \frac{L}{2} \\ H &= \frac{L}{3} \end{aligned}$$

where we use $H \neq 0$ and $W \neq 0$ to give a maximum volume.

Plug into the constraint: $\quad L + 2\left(\frac{L}{2}\right) + 3\left(\frac{L}{3}\right) = 3L = 36$

$L = 12 \qquad W = 6 \qquad H = 4 \qquad\qquad V = 12 \cdot 6 \cdot 4 = 288$

Eliminate a Variable Method:

$L = 36 - 2W - 3H$

$V = WH(36 - 2W - 3H) = 36WH - 2W^2H - 3WH^2$

$$\begin{aligned} V_W &= 36H - 4WH - 3H^2 = 0 \\ V_H &= 36W - 2W^2 - 6WH = 0 \end{aligned} \quad\Rightarrow\quad \begin{aligned} 36 - 4W - 3H &= 0 \\ 36 - 2W - 6H &= 0 \end{aligned}$$

where we use $H \neq 0$ and $W \neq 0$ to give a maximum volume.

$$\begin{aligned} 4W + 3H &= 36 \\ 2W + 6H &= 36 \end{aligned} \quad\Rightarrow\quad \begin{aligned} 8W + 6H &= 72 \\ 2W + 6H &= 36 \end{aligned} \quad\Rightarrow\quad 6W = 36 \quad\Rightarrow\quad W = 6$$

$6H = 36 - 2W = 36 - 12 = 24 \quad\Rightarrow\quad H = 4$

$L = 36 - 2W - 3H = 36 - 12 - 12 = 12 \qquad\qquad V = 12 \cdot 6 \cdot 4 = 288$

15. (10 points) Find the area of the surface $\vec{R}(p,q) = (p, p + q^2, p - q^2)$ for $0 \le p \le 3$ and $0 \le q \le 2$.

$$\begin{array}{r} \\ \vec{e}_p = \\ \vec{e}_q = \end{array} \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\ 1 & 1 & 1\ ) \\ (\ 0 & 2q & -2q\ ) \end{array} \right|$$

$\vec{N} = \vec{e}_p \times \vec{e}_q = \hat{\imath}(-2q - 2q) - \hat{\jmath}(-2q) + \hat{k}(2q) = (-4q, 2q, 2q)$

$\left|\vec{N}\right| = \sqrt{16q^2 + 4q^2 + 4q^2} = \sqrt{24}\,q$

$A = \iint 1\,dS = \iint \left|\vec{N}\right|\,dp\,dq = \int_0^2\int_0^3 \sqrt{24}\,q\,dp\,dq = \sqrt{24}\,\big[\,p\,\big]_0^3 \left[\frac{q^2}{2}\right]_0^2 = 6\sqrt{24} = 12\sqrt{6}$

16. (15 points) Compute $\iint_P \vec{\nabla}\times\vec{F}\cdot d\vec{S}$ for $\vec{F} = (-yz^2, xz^2, z^3)$ over the paraboloid $z = x^2 + y^2$ with $z \le 4$, oriented **up and in**, and parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz^2 & xz^2 & z^3 \end{vmatrix} = \hat{\imath}(0-2xz) - \hat{\jmath}(0--2yz) + \hat{k}(z^2--z^2) = (-2xz, -2yz, 2z^2)$$

$$\left(\vec{\nabla}\times\vec{F}\right)\left(\vec{R}(r,\theta)\right) = (-2r^3\cos\theta, -2r^3\sin\theta, 2r^4)$$

$$\begin{array}{r} \\ \vec{e}_r = \\ \vec{e}_\theta = \end{array} \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\ \cos\theta & \sin\theta & 2r) \\ (-r\sin\theta & r\cos\theta & 0\ ) \end{vmatrix}$$

$$\vec{N} = \vec{e}_r\times\vec{e}_\theta = \hat{\imath}(0-2r^2\cos\theta) - \hat{\jmath}(0--2r^2\sin\theta) + \hat{k}(r\cos^2\theta--r\sin^2\theta) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$\vec{N}$ is correctly oriented up and in.

$$\vec{\nabla}\times\vec{F}\cdot\vec{N} = 4r^5\cos^2\theta + 4r^5\sin^2\theta + 2r^5 = 6r^5$$

$$z = r^2 \le 4 \quad\Rightarrow\quad r \le 2$$

$$\iint_P \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^2 \vec{\nabla}\times\vec{F}\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^2 6r^5\,dr\,d\theta = 2\pi[r^6]_0^2 = 128\pi$$