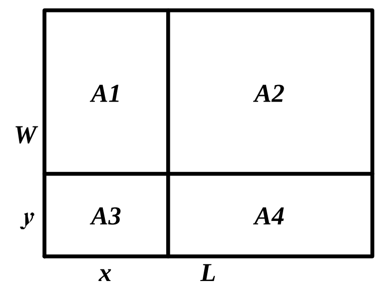
1. Minimal Rectangles and Triangles

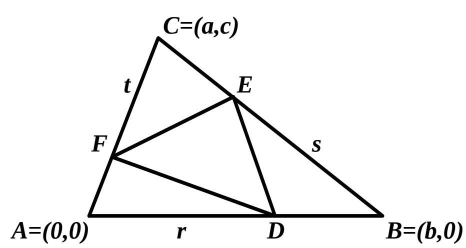
a. Consider a rectangle of length L and width W. Draw a line parallel to each side at distances x and y from one corner, as shown in the diagram:



This divides the rectangle into 4 subrectangles. Find the values of x and y which maximize and minimize the sum of the squares of the areas:

$$f = (A_1)^2 + (A_2)^2 + (A_3)^2 + (A_4)^2$$

b. Consider a triangle with vertices at $A = (0,0)$, $B = (b,0)$ and $C = (a,c)$ where a, b and c are fixed. Pick point D a fraction r of the way from A to B, point E a fraction s of the way from B to C, and point F a fraction t of the way from C to A and connect D, E and F, as shown in the diagram:



This divides the triangle into 4 subtriangles. Find the values of r, s and t which maximize and minimize the sum of the squares of the areas:

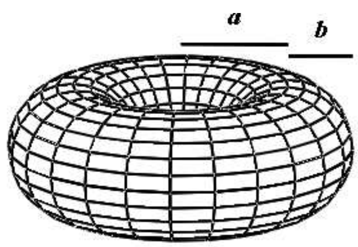
$$f = (A_1)^2 + (A_2)^2 + (A_3)^2 + (A_4)^2$$

c. In both problems, be sure to identify the configuration space and check both the interior and boundary of the configuration space for the absolute maximum and minimum.

2. Skimpy Donut

You are the mathematics consultant for a donut company which makes donuts which have a thin layer of chocolate icing covering the entire donut. One day you decide to point out that the company might cut costs on chocolate icing if they keep the volume (and hence weight) of the donut fixed but adjust the shape of the donut to minimize the surface area. Alternatively, they could advertise extra icing by maximizing the surface area. You need to write a report presenting your ideas which can be read by both the company president and the technical engineers.

A donut has the shape of a torus which is specified by giving a big radius a from the center of the hole to the center of the ring and a small radius b which is the radius of the ring, as shown in the figure.



Your job is to determine the values of a and b which extremize the surface area while keeping the volume fixed at the volume of the typical donut mentioned above. This original donut has $a = 5$ cm and $b = 3$ cm.

a. The surface of a torus satisfies the equation

$$(r - a)^2 + z^2 = b^2$$

in cylindrical coordinates where, of course, $b \le a$.

i. Compute the volume V of the torus as a function of a and b.
HINT: Integrate in cylindrical coordinates.

ii. Check that the volume of the original donut is
$V = 90\pi^2$ cm^3 ≈ 888 cm^3.

b. The surface of the torus can also be parametrized as

$$\vec{R}(\theta, \varphi) = ((a + b\cos\varphi)\cos\theta, (a + b\cos\varphi)\sin\theta, b\sin\varphi)$$

for $0 \le \theta \le 2\pi$ and $0 \le \varphi \le 2\pi$. Here, θ represents the angle around the circle of radius a and φ represents the angle around the circle of radius b.

i. Compute the surface area S of the torus as a function of a and b.
HINT: Do a surface integral in θ and φ.

ii. Check that the surface area of the original donut is
$S = 60\pi^2$ cm^2 ≈ 592 cm^2.

c. Keep the volume fixed at $V = 90\pi^2$ cm^{3}and find the values of a, b and S which minimize and maximize the surface area S. (Apply the second derivative test to any critical point in the interior and check the values at the endpoints.)

d. Write a letter to the CEO of the donut company summarizing your results (including minimum and maximum dimensions, a description of these donuts and the percent savings or extra cost). Anything you say in this report must be documented in an appendix of Maple computations for the engineers.

3. The Volume Between a Surface and Its Tangent Plane

In this project, you will be finding the tangent plane to a surface for which the volume between the surface and the tangent plane is a minimum.

a. Consider the surface

$$z = f(x,y) = 4x^2 + y^2 + x^2y^2$$

Verify that the surface is everywhere concave up on the square $0 \le x \le 1$ and $0 \le y \le 1$.

Note: A function fx,y is everywhere concave up on a region if $D = f_{xx}f_{yy} - f_{xy}{}^2 > 0$ and $f_{xx} > 0$ everywhere on the region. It is everywhere concave down on a region if $D = f_{xx}f_{yy} - f_{xy}{}^2 > 0$ and $f_{xx} < 0$ everywhere on the region.

b. Find its tangent plane at a general point $\left(a, b, f(a,b)\right)$.

c. Compute the volume between the surface and its general tangent plane above the square $0 \le x \le 1$ and $0 \le y \le 1$. Call this volume $V(a,b)$.

d. Find the point (a,b) for which the volume $V(a,b)$ is a minimum. Be sure to apply the second derivative test to verify that your critical point is a minimum.

e. Repeat steps (a)-(d) for two or three other functions $f(x,y)$. Use interesting functions, not just polynomials with at least one concave up and one concave down. Check the concavity.

f. What do you conjecture?

g. Prove your conjecture by repeating steps (a)-(d) for an undefined function $f(x,y)$.

h. What happens to your conjecture if you change the square base to another region R? Try some shapes other than a rectangle or a circle!

4. The Hypervolume of a Hypersphere

In this project, you will determine the hypervolume enclosed by a hypersphere in $\mathbb{R}^n$ whose equation is:

$$x_1{}^2 + x_2{}^2 + \cdots + x_{n-1}{}^2 + x_n{}^2 = R^2$$

a. Compute the area enclosed by the circle $x^2 + y^2 = R^2$ using a double integral in polar coordinates. Repeat using a double integral in rectangular coordinates with x as the inner integral and y as the outer integral.

Let $V_2(R)$ denote this function, where V_2 means 2-dimensional volume (area).

b. Compute the volume enclosed by the sphere $x^2 + y^2 + z^2 = R^2$ using a triple integral in spherical coordinates. Repeat using a triple integral in rectangular coordinates with x as the inner integral, y as the middle integral and z as the outer integral.

Let $V_3(R)$ denote this function, where V_3 means 3-dimensional volume.

Explain (geometrically and algebraically) why the inner 2 integrals are just $V_2\left(\sqrt{R^2 - z^2}\right)$. (Think about volume by slicing.)

c. Compute the 4-dimensional hypervolume enclosed by the hypersphere $x^2 + y^2 + z^2 + w^2 = R^2$ using a quadruple integral in rectangular coordinates with x as the inner integral, y and z as the middle integrals and w as the outer integral.

Let $V_4(R)$ denote this function, where V_4 means 4-dimensional hypervolume.

Explain (geometrically and algebraically) why the inner 3 integrals are just $V_3\left(\sqrt{R^2 - w^2}\right)$.

d. For $n = 5, 6, \cdots, 10$, find the n-dimensional hypervolume enclosed by the n-dimensional hypersphere $x_1{}^2 + x_2{}^2 + \cdots + x_{n-1}{}^2 + x_n{}^2 = R^2$.

Let $V_n(R)$ denote this function, where V_n means n-dimensional hypervolume.

HINT: If you write this volume as an n-fold integral in rectangular coordinates with x_n as the outer integral then the inner $n - 1$ integrals are $V_{n-1}\left(\sqrt{R^2 - x_n{}^2}\right)$.

Explain this in terms of volume by slicing.

e. Looking at your results for the hypervolumes of the n-dimensional hyperspheres, deduce two general patterns for $V_n(R)$. (The formulas for n even and for n odd are different.) Explain how you got your formulas. Does your "odd" formula hold for the case $n = 1$ that is, for the length of the interval $[-R, R]$?

f. Use mathematical induction to prove your two formulas for $V_n(R)$. (Use the hint from part (d) twice.) This may be hard; so get help from your instructor.