Name________________________ Sec________

1-12	/48	14	/16
13	/16	15	/25
		Total	/105

MATH 221 Final Exam Spring 2011

Section 500 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. Find the angle between the vectors $\vec{u} = (12,-3,4)$ and $\vec{v} = (2,1,-2)$.

a. $\frac{\pi}{6}$

b. $\frac{\pi}{4}$

c. $\frac{\pi}{3}$

d. $\frac{2\pi}{3}$

e. $\cos^{-1}\left(\frac{1}{3}\right)$ Correct Choice

$|\vec{u}| = \sqrt{144+9+16} = 13 \qquad |\vec{v}| = \sqrt{4+1+4} = 3 \qquad \vec{u}\cdot\vec{v} = 24-3-8 = 13$

$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{13}{13\cdot 3} = \frac{1}{3}$

2. If $\vec{u}$ points DOWN and $\vec{v}$ points NORTHEAST then $\vec{u}\times\vec{v}$ points

a. NORTHWEST

b. SOUTHWEST

c. SOUTHEAST Correct Choice

d. UP

e. WEST

Point your fingers down with the palm facing northeast, then your thumb points southeast.

3. Find the point where the lines $(x,y,z) = (3-t, 2+t, 2t)$ and $(x,y,z) = (-1+2t, 5-t, 3+t)$ intersect. At this point $x+y+z =$

a. $\frac{17}{2}$

b. 9 Correct Choice

c. $\frac{25}{3}$

d. 11

e. They do not intersect.

Change the name of the parameter in the second equation: $(x,y,z) = (-1+2s, 5-s, 3+s)$

Equate x, y and z: $3-t = -1+2s, \quad 2+t = 5-s, \quad 2t = 3+s$

Add the first 2 equations: $5 = 4+s$ So $s = 1$

Plug into the 2nd equation: $2+t = 5-1 = 4$ So $t = 2$

Check the 3rd equation is satisfied: $2t = 4$ and $3+s = 4$ OK

Plug into 1st line: $(x,y,z) = (3-t, 2+t, 2t) = (1,4,4)$

Plug into 2nd line: $(x,y,z) = (-1+2s, 5-s, 3+s) = (1,4,4)$ So $x+y+z = 9$

4. For the curve $\vec{r}(t) = (t^2, 2t, \ln t)$, compute $\hat{B} = \dfrac{\vec{v}\times\vec{a}}{|\vec{v}\times\vec{a}|}$.

a. $\left(\dfrac{-1}{2t^2+1}, \dfrac{-2t}{2t^2+1}, \dfrac{2t^2}{2t^2+1}\right)$

b. $\left(\dfrac{1}{2t^2+1}, \dfrac{-2t}{2t^2+1}, \dfrac{2t^2}{2t^2+1}\right)$

c. $\left(\dfrac{1}{2t^2+1}, \dfrac{2t}{2t^2+1}, \dfrac{2t^2}{2t^2+1}\right)$

d. $\left(\dfrac{-1}{2t^2+1}, \dfrac{-2t}{2t^2+1}, \dfrac{-2t^2}{2t^2+1}\right)$

e. $\left(\dfrac{-1}{2t^2+1}, \dfrac{2t}{2t^2+1}, \dfrac{-2t^2}{2t^2+1}\right)$ Correct Choice

$\vec{r}(t) = (t^2, 2t, \ln t)$ $\quad \vec{v} = \left(2t, 2, \dfrac{1}{t}\right)$ $\quad \vec{a} = \left(2, 0, \dfrac{-1}{t^2}\right)$ $\quad \vec{v}\times\vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2t & 2 & t^{-1} \\ 2 & 0 & -t^{-2} \end{vmatrix} = \left(\dfrac{-2}{t^2}, \dfrac{4}{t}, -4\right)$

$|\vec{v}\times\vec{a}| = \sqrt{\dfrac{4}{t^4} + \dfrac{16}{t^2} + 16} = \dfrac{2}{t^2}\sqrt{1 + 4t^2 + 4t^4} = \dfrac{2}{t^2}(1 + 2t^2)$

$\hat{B} = \dfrac{\vec{v}\times\vec{a}}{|\vec{v}\times\vec{a}|} = \dfrac{t^2}{2(1+2t^2)}\left(\dfrac{-2}{t^2}, \dfrac{4}{t}, -4\right) = \left(\dfrac{-1}{2t^2+1}, \dfrac{2t}{2t^2+1}, \dfrac{-2t^2}{2t^2+1}\right)$

5. Find the equation of the plane tangent to the graph of $z = 3x^2y - 2y^3$ at the point $(2,1)$.
The z-intercept is

a. -20 Correct Choice
b. -14
c. 14
d. 20
e. 40

$f(x,y) = 3x^2y - 2y^3$ $\quad f_x(x,y) = 6xy$ $\quad f_y(x,y) = 3x^2 - 6y^2$

$f(2,1) = 3\cdot 2^2 - 2 = 10$ $\quad f_x(2,1) = 6\cdot 2 = 12$ $\quad f_y(2,1) = 3\cdot 2^2 - 6 = 6$

$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 14 + 12(x-2) + 18(y-1)$

z-intercept $= 10 + 12(-2) + 6(-1) = -20$

6. Find the equation of the line perpendicular to the graph of $x^3y^2z - 2x^2z^2 = 10$ at the point $(1,3,2)$.
This line intersects the xy-plane at

a. $\left(-\dfrac{19}{3}, 2, 0\right)$
b. $\left(2, \dfrac{19}{3}, 0\right)$
c. $\left(\dfrac{19}{3}, -2, 0\right)$
d. $(-75, -21, 0)$ Correct Choice
e. $(21, -75, 0)$

$F(x,y,z) = x^3y^2z - 2x^2z^2$ $\quad \vec{\nabla}F = (3x^2y^2z - 4xz^2, 2x^3yz, x^3y^2 - 4x^2z)$

$\vec{N} = \vec{\nabla}F(1,3,2) = \left(3(3)^2(2) - 4(2)^2, 2(3)(2), (3)^2 - 4(2)\right) = (38, 12, 1)$

$X = P + t\vec{N}$ $\quad (x,y,z) = (1,3,2) + t(38,12,1) = (1+38t, 3+12t, 2+t)$

This line intersects the xy-plane when $z = 2 + t = 0$ or $t = -2$ or at $(x,y,z) = (-75, -21, 0)$

7. Han Deut is flying the Millennium Eagle through a dangerous zenithon field whose density is $\rho = xyz$. If his current position is $\vec{r} = (2,1,-1)$, and his current velocity is $\vec{v} = (3,2,1)$ find the current rate of change of the density.

a. -5 Correct Choice

b. -1

c. 1

d. 3

e. 18

$\vec{\nabla}\rho = (yz, xz, xy) \qquad \vec{\nabla}\rho\Big|_{(2,1,-1)} = (-1,-2,2)$

$\dfrac{d\rho}{dt} = \vec{v} \cdot \vec{\nabla}\rho = (3,2,1) \cdot (-1,-2,2) = -5$

8. Compute $\iint xy\,dA$ over the region between $y = x^2$ and $y = 3x$.

a. $\dfrac{3^3}{4}$

b. $\dfrac{3^4}{5}$

c. $\dfrac{3^5}{4}$

d. $\dfrac{3^5}{8}$ Correct Choice

e. $\dfrac{3^6}{2}$

The curves intersect when $x^2 = 3x$ or $x = 0,3$.

$$\iint xy\,dA = \int_0^3 \int_{x^2}^{3x} xy\,dy\,dx = \int_0^3 x\left[\frac{y^2}{2}\right]_{y=x^2}^{3x} dx = \frac{1}{2}\int_0^3 (9x^3 - x^5)\,dx = \frac{1}{2}\left[\frac{9x^4}{4} - \frac{x^6}{6}\right]_0^3$$

$$= \frac{3^6}{2}\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{3^6}{24} = \frac{3^5}{8}$$

9. Compute $\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} \cos(x^2+y^2)\,dx\,dy$.

HINT: Switch to polar coordinates.

a. $\pi\cos(4)$

b. $\pi\cos(4) - \pi$

c. $\dfrac{\pi}{2}\sin(4)$ Correct Choice

d. $\dfrac{\pi}{2}\sin(4) - \dfrac{\pi}{2}$

e. $2\pi\cos(4) - 2\pi$

This is the right half of the circle $x^2 + y^2 = 4$.

$$\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} \cos(x^2+y^2)\,dx\,dy = \int_{-\pi/2}^{\pi/2}\int_0^2 \cos(r^2)\,r\,dr\,d\theta = \pi\frac{\sin(r^2)}{2}\Bigg|_0^2 = \frac{\pi}{2}\sin(4)$$

10. Compute $\iiint z\,dV$ over the volume above the paraboloid $z = x^2 + y^2$ below $z = 4$.

a. $\dfrac{64\pi}{3}$ Correct Choice

b. $\dfrac{32\pi}{3}$

c. $\dfrac{16\pi}{5}$

d. 21π

e. 11π

$z = r^2 \qquad r^2 = 4 \quad \Rightarrow \quad r = 2$

$$\iiint z\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 z\,r\,dz\,dr\,d\theta = 2\pi\int_0^2\left[\frac{z^2}{2}\right]_{z=r^2}^4 r\,dr = 2\pi\int_0^2\left(\frac{16}{2} - \frac{r^4}{2}\right)r\,dr = \pi\int_0^2(16r - r^5)\,dr$$

$$= \pi\left[16\frac{r^2}{2} - \frac{r^6}{6}\right]_0^2 = \pi\left(32 - \frac{32}{3}\right) = \frac{64\pi}{3}$$

11. Compute $\int \vec{F}\cdot d\vec{s}$ for $\vec{F} = (y+z, x+z, x+y)$ along the curve $\vec{r}(t) = \left(2\dfrac{t^2+2}{t+1}, 3\dfrac{t^2+4}{t+2}, 4\dfrac{t^2+6}{t+3}\right)$ from $(4,6,8)$ to $(3,5,7)$.

HINT: Find a scalar potential.

a. 33

b. 6

c. 0

d. -6

e. -33 Correct Choice

$\vec{F} = \vec{\nabla} f \qquad \partial_x f = y+z \qquad \partial_y f = x+z \qquad \partial_z f = x+y \quad \Rightarrow \quad f(x,y,z) = xy + xz + yz$

$$\int \vec{F}\cdot d\vec{s} = \int \vec{\nabla} f\cdot d\vec{s} = f(3,5,7) - f(4,6,8) = (3\cdot5 + 3\cdot7 + 5\cdot7) - (4\cdot6 + 4\cdot8 + 6\cdot8) = 71 - 104 = -33$$

12. Compute $\oint \vec{F}\cdot d\vec{s}$ for $\vec{F} = (\sin(x^3) - 4y, \tan(y^5) + 6x)$ counterclockwise around the triangle with vertices $(0,0)$, $(2,0)$ and $(0,6)$.

Hint: Use Green's Theorem.

a. 6

b. 12

c. 30

d. 60 Correct Choice

e. 80

$P = \sin(x^3) - 4y \qquad Q = \tan(y^5) + 6x \qquad \partial_x Q - \partial_y P = 6 - (-4) = 10$

$$\oint \vec{F}\cdot d\vec{s} = \oint P\,dx + Q\,dy = \iint \partial_x Q - \partial_y P\,dx\,dy = \iint 10\,dx\,dy = 10\text{Area} = 10\cdot\frac{1}{2}\cdot 2\cdot 6 = 60$$

Work Out: (Points indicated. Part credit possible. Show all work.)

13. (16 points) Use Lagrange multipliers to find 4 numbers, a, b, c, and d, for which $a+2b+3c+4d=48$ and whose product is a maximum.

Maximize $f=abcd$ subject to the constraint $g=a+2b+3c+4d=48$.

$\vec{\nabla}f=(bcd,acd,abd,abc)$ $\quad\vec{\nabla}g=(1,2,3,4)$

$\vec{\nabla}f=\lambda\vec{\nabla}g$: $\quad bcd=\lambda \quad acd=2\lambda \quad abd=3\lambda \quad abc=4\lambda$

Make the left sides all be $abcd$ and equate: $\quad abcd=\lambda a=2\lambda b=3\lambda c=4\lambda d$

So $\quad b=\frac{a}{2} \quad c=\frac{a}{3} \quad d=\frac{a}{4}$ $\quad$ Substitute into the constraint:

$a+2b+3c+4d=a+a+a+a=48 \quad\Rightarrow\quad a=12,\ b=6,\ c=4,\ d=3$

14. (16 points) Find the mass and the y-component of the center of mass of the quarter of the sphere $x^2+y^2+z^2\le 4$ in the first octant $(x\ge 0, y\ge 0, z\ge 0)$ if the mass density is $\delta=xyz$.

$$\delta=xyz=\rho\sin\varphi\cos\theta\rho\sin\varphi\sin\theta\rho\cos\varphi=\rho^3\sin\theta\cos\theta\sin^2\varphi\cos\varphi \qquad dV=\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$M=\iiint\delta\,dV=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2\rho^5\sin\theta\cos\theta\sin^3\varphi\cos\varphi\,d\rho\,d\theta\,d\varphi=\left[\frac{\rho^6}{6}\right]_0^2\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}\left[\frac{\sin^4\varphi}{4}\right]_0^{\pi/2}$$

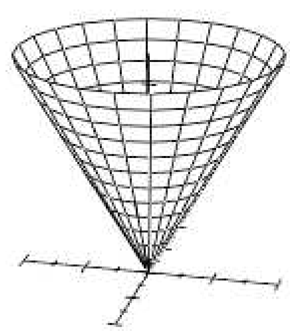
$$=\frac{2^6}{6}\frac{1}{2}\frac{1}{4}=\frac{4}{3}$$

$$M_{xz}=\iiint y\delta\,dV=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2\rho^6\sin^2\theta\cos\theta\sin^4\varphi\cos\varphi\,d\rho\,d\theta\,d\varphi=\left[\frac{\rho^7}{7}\right]_0^2\left[\frac{\sin^3\theta}{3}\right]_0^{\pi/2}\left[\frac{\sin^5\varphi}{5}\right]_0^{\pi/2}$$

$$=\frac{2^7}{7}\frac{1}{3}\frac{1}{5}=\frac{128}{105}$$

$$\bar{y}=\frac{M_{xz}}{M}=\frac{128}{105}\frac{3}{4}=\frac{32}{35}$$

15. (25 points) Verify Stokes' Theorem $\iint\limits_C \vec{\nabla}\times\vec{F}\cdot d\vec{S}=\oint\limits_{\partial C}\vec{F}\cdot d\vec{s}$



for the vector field $\vec{F}=(yz,-xz,z^2)$ and the cone

$z=2\sqrt{x^2+y^2}\le 2$ oriented down and out.

Be careful with orientations. Use the following steps:

First the Left Hand Side:

a. Compute the curl:

$$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ yz & -xz & z^2\end{vmatrix}=\hat{\imath}(--x)-\hat{\jmath}(-y)+\hat{k}(-z-z)=(x,y,-2z)$$

b. Complete the parametrization of the surface C:

$\vec{R}(r,\theta)=\left(r\cos\theta, r\sin\theta, \underline{\quad 2r \quad}\right)$

c. Compute the tangent vectors:

$\vec{e}_r=(\cos\theta,\sin\theta,2)$

$\vec{e}_\theta=(-r\sin\theta,r\cos\theta,0)$

d. Compute the normal vector:

$$\vec{N} = \hat{\imath}(-2r\cos\theta) - \hat{\jmath}(- -2r\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (-2r\cos\theta, -2r\sin\theta, r)$$

This is up and in. Reverse $\vec{N} = (2r\cos\theta, 2r\sin\theta, -r)$

e. Evaluate $\vec{\nabla} \times \vec{F}$ on the surface:

$$\vec{\nabla} \times \vec{F}\Big|_{\vec{R}(r,\theta)} = (r\cos\theta, r\sin\theta, -4r)$$

f. Compute the dot product:

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = 2r^2\cos^2\theta + 2r^2\sin^2\theta + 4r^2 = 6r^2$$

g. Find the limits of integration:

$0 \le \theta \le 2\pi \qquad 2\sqrt{x^2 + y^2} = 2$ when $r = 1$

h. Compute the left hand side:

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^1 6r^2\,dr\,d\theta = 12\pi\left[\frac{r^3}{3}\right]_0^1 = 4\pi$$

Second the Right Hand Side:

i. Parametrize the boundary circle ∂C:

$$\vec{r}(\theta) = (\cos\theta, \sin\theta, 2)$$

j. Compute the tangent vector:

$$\vec{v} = (-\sin\theta, \cos\theta, 0)$$

This is CCW. Reverse $\vec{v} = (\sin\theta, -\cos\theta, 0)$

k. Evaluate $\vec{F} = (yz, -xz, z^2)$ on the curve:

$$\vec{F}\Big|_{\vec{r}(\theta)} = (2\sin\theta, -2\cos\theta, 4)$$

l. Compute the dot product:

$$\vec{F} \cdot \vec{v} = 2\sin^2\theta + 2\cos^2\theta = 2$$

m. Compute the right hand side:

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} 2\,d\theta = 4\pi$$