Line & Surface Integral Notation

PARAMETRIZED CURVES & LINE INTEGRALS:

Curve:

$$\vec{r}(t) = (x(t), y(t), z(t))$$

Tangent vector:

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

Tangent differential vector or Vector differential of arc length:

$$d\vec{s} = d\vec{r} = (dx, dy, dz) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)dt = \vec{v}\,dt = \hat{v}|\vec{v}|\,dt = \hat{v}\,ds$$

Tangent differential scalar or Scalar differential of arc length:

$$ds = |d\vec{s}| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\;dt = |\vec{v}|\,dt$$

Arc length integral:

$$L = \int_A^B ds = \int_a^b |\vec{v}|\,dt$$

Integral of a scalar $f(x, y, z)$ along $\vec{r}(t)$ from $A = \vec{r}(a)$ to $B = \vec{r}(b)$:

$$\int_A^B f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{v}|\,dt$$

Average value of a function $f(x, y, z)$ along $\vec{r}(t)$ from $A = \vec{r}(a)$ to $B = \vec{r}(b)$:

$$f_{\text{ave}} = \frac{1}{L}\int_A^B f\,ds = \frac{1}{L}\int_a^b f(\vec{r}(t))\,|\vec{v}|\,dt$$

Total mass: Center of mass:

$$M = \int_A^B \rho\,ds = \int_a^b \rho\,|\vec{v}|\,dt \qquad\qquad (\bar{x}, \bar{y}, \bar{z}) = \frac{1}{M}\int_A^B (x, y, z)\,\rho\,ds$$

Integral of a vector field $\vec{F} = (F_1, F_2, F_3)$ along $\vec{r}(t)$ from $A = \vec{r}(a)$ to $B = \vec{r}(b)$ or Work:

$$\int_A^B \vec{F}\cdot d\vec{s} = \int_A^B (F_1\,dx + F_2\,dy + F_3\,dz) = \int_a^b \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt = \int_a^b \vec{F}\cdot\vec{v}\,dt = \int_A^B \vec{F}\cdot\hat{v}\,ds$$

SPECIAL FOR CURVES IN R^2:

Normal vector:

$$\vec{n} = \vec{v}^{\perp} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} \\ v_1 & v_2 \end{vmatrix} = v_2\hat{\imath} - v_1\hat{\jmath} = \frac{dy}{dt}\hat{\imath} - \frac{dx}{dt}\hat{\jmath} \qquad |\vec{n}| = |\vec{v}|$$

Normal differential vector:

$$d\vec{n} = dy\,\hat{\imath} - dx\,\hat{\jmath} = \left(\frac{dy}{dt}\hat{\imath} - \frac{dx}{dt}\hat{\jmath}\right)dt = \vec{n}\,dt = \hat{n}|\vec{n}|\,dt = \hat{n}\,ds$$

Normal differential scalar:

$$dn = |d\vec{n}| = \sqrt{(dy)^2 + (dx)^2} = ds$$

Integral of the normal component of vector field $\vec{G} = G_1\hat{\imath} + G_2\hat{\jmath}$ along $\vec{r}(t)$:

$$\int_A^B \vec{G}\cdot d\vec{n} = \int_A^B (G_1\,dy - G_2\,dx) = \int_a^b \left(G_1\frac{dy}{dt} - G_2\frac{dx}{dt}\right)dt = \int_a^b \vec{G}\cdot\vec{n}\,dt = \int_A^B \vec{G}\cdot\hat{n}\,ds$$

Further, if $\quad \vec{G} = \vec{F}^{\perp} = F_2\hat{\imath} - F_1\hat{\jmath} \quad$ then:

$$\vec{G}\cdot\vec{n} = (F_2, -F_1)\cdot(v_2, -v_1) = \vec{F}\cdot\vec{v}$$

and

$$\int_A^B \vec{G}\cdot d\vec{n} = \int_A^B (F_2\,dy - (-F_1)\,dx) = \int_A^B \vec{F}\cdot d\vec{s}$$

PARAMETRIZED SURFACES & SURFACE INTEGRALS:

Surface:

$$\vec{R}(s,t) = (x(s,t), y(s,t), z(s,t))$$

Tangent vectors:

$$\vec{e}_s = \frac{\partial \vec{R}}{\partial s} = \left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}\right) \quad \text{and} \quad \vec{e}_t = \frac{\partial \vec{R}}{\partial t} = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right)$$

Normal vector:

$$\vec{N} = \vec{e}_s \times \vec{e}_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} & \dfrac{\partial z}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \end{vmatrix} = \frac{\partial(y,z)}{\partial(s,t)}\hat{i} + \frac{\partial(z,x)}{\partial(s,t)}\hat{j} + \frac{\partial(x,y)}{\partial(s,t)}\hat{k}$$

Surface differential vector or Vector differential of surface area:

$$d\vec{S} = (dy\,dz, dz\,dx, dx\,dy) = \left(\frac{\partial(y,z)}{\partial(s,t)}, \frac{\partial(z,x)}{\partial(s,t)}, \frac{\partial(x,y)}{\partial(s,t)}\right) ds\,dt$$

$$= \vec{N}\,ds\,dt = \hat{N}|\vec{N}|\,ds\,dt = \hat{N}\,dS$$

Surface differential scalar or Scalar differential of surface area:

$$dS = |d\vec{S}| = \sqrt{(dy\,dz)^2 + (dz\,dx)^2 + (dx\,dy)^2} = \sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2}\, ds\,dt$$

$$= |\vec{N}|\,ds\,dt$$

Surface area integral:

$$A = \iint_{\vec{R}} dS = \iint_{\vec{R}} |\vec{N}|\,ds\,dt$$

Integral of a scalar $f(x,y,z)$ over $\vec{R}(s,t)$:

$$\iint_{\vec{R}} f\,dS = \iint_{\vec{R}} f(\vec{R}(s,t))|\vec{N}|\,ds\,dt$$

Average value of a function $f(x,y,z)$ over $\vec{R}(s,t)$:

$$f_{ave} = \frac{1}{A}\iint_{\vec{R}} f\,dS = \frac{1}{A}\iint_{\vec{R}} f(\vec{R}(s,t))|\vec{N}|\,ds\,dt$$

Total mass: Center of mass:

$$M = \iint_{\vec{R}} \rho\,dS = \iint_{\vec{R}} \rho|\vec{N}|\,ds\,dt \qquad\qquad (\bar{x},\bar{y},\bar{z}) = \frac{1}{M}\iint_{\vec{R}}(x,y,z)\rho\,dS$$

Integral of a vector field $\vec{F} = (F_1, F_2, F_3)$ over $\vec{R}(u,v)$ or Flux:

$$\iint_{\vec{R}} \vec{F}\cdot d\vec{S} = \iint_{\vec{R}}(F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iint_{\vec{R}}\left(F_1\frac{\partial(y,z)}{\partial(s,t)} + F_2\frac{\partial(z,x)}{\partial(s,t)} + F_3\frac{\partial(x,y)}{\partial(s,t)}\right) ds\,dt$$

$$= \iint_{\vec{R}} \vec{F}\cdot\vec{N}\,ds\,dt = \iint_{\vec{R}} \vec{F}\cdot\hat{N}\,dS$$