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10	/15
11	/25
12	/15
Total	/100

MATH 221 Final Exam Fall 2012

Sections 502 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. If $\vec{a} = (3,3,-1)$ and $\vec{b} = (2,5,1)$ then $\left|\vec{a} - 3\vec{b}\right| =$

a. 26

b. 13 Correct Choice

c. 0

d. -13

e. -26

SOLUTION:

$\vec{a} - 3\vec{b} = (3,3,-1) - 3(2,5,1) = (3-6, 3-15, -1-3) = (-3,-12,-4)$

$\left|\vec{a} - 3\vec{b}\right| = \sqrt{9+144+16} = \sqrt{169} = 13$

2. Find the point on the elliptic paraboloid $z - \dfrac{x^2}{2} - \dfrac{y^2}{4} = 1$ where a unit normal is $\left(\dfrac{2}{3}, \dfrac{1}{3}, -\dfrac{2}{3}\right)$.

a. $(-2,-4,7)$

b. $(2,4,7)$

c. $(4,2,10)$

d. $\left(1,1,\dfrac{7}{4}\right)$ Correct Choice

e. $\left(-1,-1,\dfrac{7}{4}\right)$

SOLUTION: $f = z - \dfrac{x^2}{2} - \dfrac{y^2}{4}$ $\quad \vec{\nabla} f = \left(-x, -\dfrac{y}{2}, 1\right) = k\left(\dfrac{2}{3}, \dfrac{1}{3}, -\dfrac{2}{3}\right)$

$-x = k\dfrac{2}{3}, \quad -\dfrac{y}{2} = k\dfrac{1}{3}, \quad 1 = -k\dfrac{2}{3} \quad k = -\dfrac{3}{2} \quad x = -k\dfrac{2}{3} = 1 \quad y = -k\dfrac{2}{3} = 1$

$z = 1 + \dfrac{x^2}{2} + \dfrac{y^2}{4} = 1 + \dfrac{1}{2} + \dfrac{1}{4} = \dfrac{7}{4}$

3. Find the point on the curve $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ where the unit tangent is $\hat{T} = \left(\frac{1}{5}, \frac{2\sqrt{2}}{5}, \frac{4}{5}\right)$.

a. $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{12}\right)$

b. $\left(1, 1, \frac{2}{3}\right)$

c. $\left(\sqrt{2}, 2, \frac{4}{3}\sqrt{2}\right)$ Correct Choice

d. $\left(2, 4, \frac{16}{3}\right)$

e. $\left(-1, 1, -\frac{2}{3}\right)$

SOLUTION: $\vec{v}(t) = (1, 2t, 2t^2)$ $|\vec{v}| = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2$

$\hat{T} = \left(\frac{1}{1+2t^2}, \frac{2t}{1+2t^2}, \frac{2t^2}{1+2t^2}\right) = \left(\frac{1}{5}, \frac{2\sqrt{2}}{5}, \frac{4}{5}\right)$ $1 + 2t^2 = 5$ $t = \sqrt{2}$

$\vec{r}\left(\sqrt{2}\right) = \left(\sqrt{2}, 2, \frac{4}{3}\sqrt{2}\right)$

4. Compute $\int_0^8 \int_{y^{1/3}}^2 \cos(x^4)\,dx\,dy$

HINT: Plot the region of integration and reverse the order of integration.

a. $\frac{1}{4}\sin(64)$

b. $\frac{1}{4}\cos(64)$

c. $\frac{1}{4}\sin(16) - \frac{1}{4}$

d. $\frac{1}{4}\cos(16) - \frac{1}{4}$

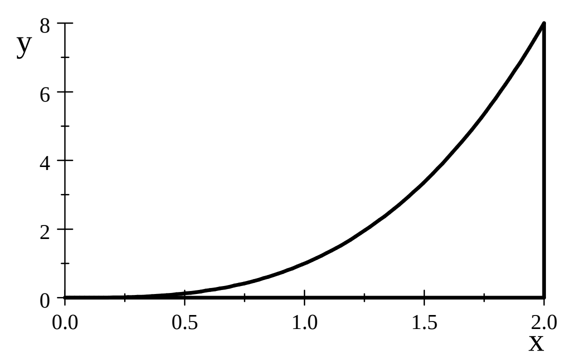
e. $\frac{1}{4}\sin(16)$ Correct Choice

SOLUTION:

$$0 \le y \le 8$$
$$y^{1/3} \le x \le 2$$

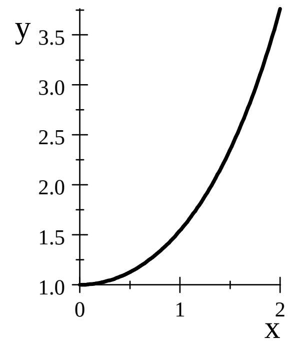
or

$$0 \le x \le 2$$
$$0 \le y \le x^3$$



$$\int_0^8 \int_{y^{1/3}}^2 \cos(x^4)\,dx\,dy = \int_0^2 \int_0^{x^3} \cos(x^4)\,dy\,dx = \int_0^2 \left[y\cos(x^4)\right]_0^{x^3} dx$$
$$= \int_0^2 x^3 \cos(x^4)\,dx = \left[\frac{1}{4}\sin(x^4)\right]_0^2 = \frac{1}{4}\sin(16)$$

5. Find the y-component of the centroid (center of mass with density 1) of the hyperbolic cosine curve $y = \cosh x$ for $0 \le x \le 2$.



HINTS: Parametrize the curve as $\vec{r}(t) = (t, \cosh t)$.

$\cosh x = \dfrac{e^x + e^{-x}}{2}$ $\quad \sinh x = \dfrac{e^x - e^{-x}}{2}$ $\quad \cosh^2 x - \sinh^2 x = 1$

$\sinh 0 = 0 \quad \cosh 0 = 1 \quad (\sinh x)' = \cosh x \quad (\cosh x)' = \sinh x$

$\sinh^2 x = \dfrac{\cosh 2x - 1}{2} \quad \cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

a. $\bar{y} = \dfrac{4 + \sinh 4}{4 \sinh 2}$ Correct Choice

b. $\bar{y} = \dfrac{1 + \cosh 4}{2 \sinh 2}$

c. $\bar{y} = \dfrac{4 + \cosh 4}{2 \sinh 2}$

d. $\bar{y} = \dfrac{\cosh 4 - 1}{2 \sinh 2}$

e. $\bar{y} = \dfrac{1 + \sinh 4}{2 \sinh 2}$

SOLUTION: $\vec{v} = (1, \sinh t) \quad |\vec{v}| = \sqrt{1 + \sinh^2 t} = \cosh t$

$$L = \int ds = \int_0^2 |\vec{v}|\,dt = \int_0^2 \cosh t\,dt = \Big[\sinh t\Big]_0^2 = \sinh 2$$

$$L_x = \int y\,ds = \int_0^2 y|\vec{v}|\,dt = \int_0^2 \cosh^2 t\,dt = \int_0^2 \frac{\cosh 2t + 1}{2}\,dt = \frac{1}{2}\left[\frac{\sinh 2t}{2} + t\right]_0^2 = \frac{\sinh 4}{4} + 1$$

$$\bar{y} = \frac{Lx}{L} = \frac{\dfrac{\sinh 4}{4} + 1}{\sinh 2} = \frac{4 + \sinh 4}{4 \sinh 2}$$

6. Find the line perpendicular to the surface $y^2 z - 2xz = 6$ at the point $(1,2,4)$. This line intersects the xy-plane at

a. $(-17, 30, 0)$

b. $(17, -30, 0)$ Correct Choice

c. $(-15, 34, 0)$

d. $(15, -34, 0)$

e. The line does not intersect the xy-plane.

SOLUTION: $f = y^2 z - 2xz \quad \vec{\nabla} f = (-2z, 2yz, y^2 - 2x)$

The direction of the line is the normal to the plane at $P = (1,2,4)$:

$\vec{v} = \vec{N} = \vec{\nabla} f\Big|_{(1,2,4)} = (-8, 16, 2)$

Line: $\vec{r}(t) = P + t\vec{v} = (1,2,4) + t(-8,16,2) = (1 - 8t, 2 + 16t, 4 + 2t)$

This line intersects the xy-plane when $z = 0 = 4 + 2t$. So $t = -2$.

at the point $\vec{r}(-2) = (1 - 8(-2), 2 + 16(-2), 4 + 2(-2)) = (17, -30, 0)$

7. Let $L = \lim_{(x,y)\to(0,0)} \frac{e^{\left(x^2+y^2\right)} - 1}{x^2+y^2}$

- **a.** L does not exist by looking at the paths $y = x$ and $y = -x$.
- **b.** L exists and $L = 1$ by looking at the paths $y = mx$.
- **c.** L does not exist by looking at polar coordinates.
- **d.** L exists and $L = 1$ by looking at polar coordinates. Correct Choice
- **e.** L exists and $L = 0$ by looking at polar coordinates.

SOLUTION: Along $y = mx$, we have $L = \lim_{x\to 0} \frac{e^{\left(1+m^2\right)x^2} - 1}{(1+m^2)x^2} \overset{\text{l'H}}{=} \lim_{x\to 0} \frac{e^{\left(1+m^2\right)x^2}(1+m^2)2x}{(1+m^2)2x} = 1,$

for all m including 1 and -1 which proves nothing.

In polar coordinates, $L = \lim_{r\to 0} \frac{e^{r^2} - 1}{r^2} \overset{\text{l'H}}{=} \lim_{r\to 0} \frac{e^{r^2} 2r}{2r} = 1,$ which proves the limit exists and $= 1$.

8. Compute $\oint \vec{F} \cdot d\vec{s}$ for $\vec{F} = (-x^2y, xy^2)$ along the semicircle $y = \sqrt{16 - x^2}$ from $(4,0)$ to $(-4,0)$ followed by the line segment from $(-4,0)$ back to $(4,0)$.
HINT: Use Green's Theorem.

- **a.** 0
- **b.** $\frac{64}{5}\pi$
- **c.** $\frac{64}{3}\pi$
- **d.** 32π
- **e.** 64π Correct Choice

SOLUTION: $\oint \vec{F} \cdot d\vec{s} = \oint P\,dx + Q\,dy$ with $P = -x^2y$ and $Q = xy^2$. By Green's Theorem,

$$\oint \vec{F} \cdot d\vec{s} = \int_{-4}^{4} \int_{0}^{\sqrt{4-x^2}} (\partial_x Q - \partial_y P)\,dy\,dx = \int_{-4}^{4} \int_{0}^{\sqrt{4-x^2}} (y^2 + x^2)\,dy\,dx = \int_0^{\pi} \int_0^4 r^2\,r\,dr\,d\theta = \pi\left[\frac{r^4}{4}\right]_0^4 = 64\pi$$

9. Compute $\oint \vec{F} \cdot d\vec{s}$ for $\vec{F} = (2xy + y^2, x^2 + 2xy)$ along the parabola $y = x^2$ from $(1,1)$ to $(2,4)$.
HINT: Find a scalar potential.

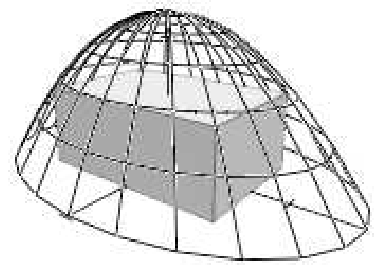
- **a.** 11
- **b.** 32
- **c.** 46 Correct Choice
- **d.** 50
- **e.** 92

SOLUTION:
$\vec{F} = (2xy + y^2, x^2 + 2xy) = \vec{\nabla} f$ for $f = x^2y + xy^2$ since $\partial_x f = 2xy + y^2$ and $\partial_y f = x^2 + 2xy$

By the F.T.C.C. $\int \vec{F} \cdot d\vec{s} = \int_{(1,1)}^{(2,4)} \vec{\nabla} f \cdot d\vec{s} = f(2,4) - f(1,1) = (2)^2(4) + (2)(4)^2 - (1)^2(1) - (1)(1)^2 = 46$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (15 points) A rectangle sits on the xy-plane with its top 4 vertices on the elliptic paraboloid $\frac{x^2}{4} + \frac{y^2}{9} + z = 6$. Find the dimensions and volume of the largest such box.



$L = 2x \quad W = 2y \quad H = z \quad$ Maximize $V = LWH = 4xyz$

METHOD 1: Lagrange Multipliers:

$V = LWH = 4xyz \qquad \vec{\nabla}V = (4yz, 4xz, 4xy)$

$g = \frac{x^2}{4} + \frac{y^2}{9} + z \qquad \vec{\nabla}g = \left(2\frac{x}{4}, 2\frac{y}{9}, 1\right)$

$\vec{\nabla}V = \lambda\vec{\nabla}g \qquad 4yz = \lambda 2\frac{x}{4} \qquad 4xz = \lambda 2\frac{y}{9} \qquad 4xy = \lambda$

$4xyz = \lambda 2\frac{x^2}{4} = \lambda 2\frac{y^2}{9} = \lambda z \qquad \Rightarrow \qquad \frac{x^2}{4} = \frac{y^2}{9} = \frac{z}{2}$

Constraint: $\quad 6 = \frac{x^2}{4} + \frac{y^2}{9} + z = \frac{z}{2} + \frac{z}{2} + z = 2z \qquad \Rightarrow \qquad z = 3$

$\frac{x^2}{4} = \frac{z}{2} = \frac{3}{2} \qquad \Rightarrow \qquad x = \sqrt{6} \qquad \frac{y^2}{9} = \frac{z}{2} = \frac{3}{2} \qquad \Rightarrow \qquad y = \frac{3}{2}\sqrt{6}$

Dimensions are $\quad L = 2x = 2\sqrt{6} \qquad W = 2y = 3\sqrt{6} \qquad H = z = 3$

Volume: $\quad V = 2\sqrt{6}\,3\sqrt{6}\,3 = 108$

METHOD 2: Eliminate a Variable:

$z = 6 - \frac{x^2}{4} - \frac{y^2}{9} \qquad V = LWH = 4xyz = 4xy\left(6 - \frac{x^2}{4} - \frac{y^2}{9}\right) = 24xy - x^3y - \frac{4}{9}xy^3$

$V_x = 24y - 3x^2y - \frac{4}{9}y^3 = y\left(24 - 3x^2 - \frac{4}{9}y^2\right) = 0 \qquad y \neq 0 \qquad 24 - 3x^2 - \frac{4}{9}y^2 = 0 \qquad (1)$

$V_y = 24x - x^3 - \frac{4}{3}xy^2 = x\left(24 - x^2 - \frac{4}{3}y^2\right) = 0 \qquad x \neq 0 \qquad 24 - x^2 - \frac{4}{3}y^2 = 0 \qquad (2)$

3(1)-(2): $\quad 48 - 8x^2 = 0 \qquad \Rightarrow \qquad x^2 = 6 \qquad \Rightarrow \qquad x = \sqrt{6}$

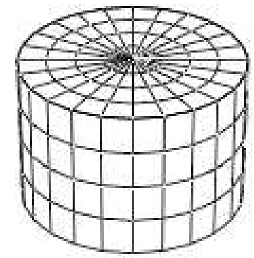
3(2)-(1): $\quad 48 - 4y^2 + \frac{4}{9}y^2 = 0 \qquad \Rightarrow \qquad 32y^2 = 48 \cdot 9 \qquad \Rightarrow \qquad y = \frac{3}{2}\sqrt{6}$

$z = 6 - \frac{x^2}{4} - \frac{y^2}{9} = 6 - \frac{6}{4} - \frac{48}{32} = 3$

Dimensions are $\quad L = 2x = 2\sqrt{6} \qquad W = 2y = 3\sqrt{6} \qquad H = z = 3$

Volume: $\quad V = 2\sqrt{6}\,3\sqrt{6}\,3 = 108$

11. (25 points) Verify Gauss' Theorem $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$



for the vector field $\vec{F} = (xz^2, yz^2, z(x^2+y^2))$ and the solid cylinder $x^2+y^2 \le 4$ for $-3 \le z \le 3$.

Be careful with orientations. Use the following steps:

First the Left Hand Side:

a. Compute the divergence:

$\vec{\nabla} \cdot \vec{F} = z^2 + z^2 + x^2 + y^2 = x^2 + y^2 + 2z^2$

b. Express the divergence and the volume element in the appropriate coordinate system:

$\vec{\nabla} \cdot \vec{F} = r^2 + 2z^2 \qquad\qquad dV = r\,dr\,d\theta\,dz$

c. Compute the left hand side:

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \int_{-3}^{3}\int_0^{2\pi}\int_0^2 (r^2+2z^2)\,r\,dr\,d\theta\,dz = 2\pi\int_{-3}^{3}\left[\frac{r^4}{4} + z^2r^2\right]_0^2 dz = 2\pi\int_{-3}^{3}(4+4z^2)\,dz$$

$$= 2\pi\left[4z + \frac{4z^3}{3}\right]_{-3}^{3} = 4\pi(12+36) = 192\pi$$

Second the Right Hand Side:

The boundary surface consists of the cylindrical sides C, a disk T at the top and a disk B at the bottom with appropriate orientations.

d. Parametrize the cylinder C:

$\vec{R}(\theta, z) = (2\cos\theta, 2\sin\theta, z)$

e. Compute the tangent vectors:

$\vec{e}_\theta = (-2\sin\theta,\ 2\cos\theta,\ 0)$

$\vec{e}_z = (\ 0,\ 0,\ 1)$

f. Compute the normal vector:

$\vec{N} = \hat{\imath}(2\cos\theta) - \hat{\jmath}(-2\sin\theta) + \hat{k}(0) = (2\cos\theta, 2\sin\theta, 0)$

This is oriented correctly outward.

g. Evaluate $\vec{F} = (xz^2, yz^2, z(x^2+y^2))$ on the cylinder:

$\vec{F}\Big|_{\vec{R}(r,\theta)} = (2\cos\theta\, z^2, 2\sin\theta\, z^2, 4z)$

h. Compute the dot product:

$\vec{F} \cdot \vec{N} = 4\cos^2\theta\, z^2 + 4\sin^2\theta\, z^2 = 4z^2$

i. Compute the flux through C:

$$\iint_C \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_{-3}^{3} 4z^2\,dz\,d\theta = 2\pi\left[\frac{4z^3}{3}\right]_{-3}^{3} = 144\pi$$

j. Parametrize the **TOP** disk T:

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 3)$

k. Compute the tangent vectors:

$\vec{e}_r = (\quad \cos\theta, \qquad \sin\theta, \qquad 0)$

$\vec{e}_\theta = (-r\sin\theta, \qquad r\cos\theta, \qquad 0)$

l. Compute the normal vector:

$\vec{N} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (0,0,r)$

This is oriented correctly upward.

m. Evaluate $\vec{F} = (xz^2, yz^2, z(x^2+y^2))$ on the top disk:

$\vec{F}\Big|_{\vec{R}(\theta,\varphi)} = (9r\cos\theta, 9r\sin\theta, 3r^2)$

n. Compute the dot product:

$\vec{F}\bullet\vec{N} = 0 + 0 + r^4 = 3r^3$

o. Compute the flux through T:

$$\iint_T \vec{F}\bullet d\vec{S} = \int_0^{2\pi}\int_0^2 3r^3\,dr\,d\theta = 2\pi\left[\frac{3r^4}{4}\right]_0^2 = 24\pi$$

p. Parametrize the **BOTTOM** disk B:

$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, -3)$

q. Compute the tangent vectors:

$\vec{e}_r = (\quad \cos\theta, \qquad \sin\theta, \qquad 0)$

$\vec{e}_\theta = (-r\sin\theta, \qquad r\cos\theta, \qquad 0)$

r. Compute the normal vector:

$\vec{N} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (0,0,r)$

This is upward. Need down. Reverse: $\vec{N} = (0,0,-r)$

s. Evaluate $\vec{F} = (xz^2, yz^2, z(x^2+y^2))$ on the bottom disk:

$\vec{F}\Big|_{\vec{R}(\theta,\varphi)} = (9r\cos\theta, 9r\sin\theta, -3r^2)$

t. Compute the dot product:

$\vec{F}\bullet\vec{N} = 3r^3$

u. Compute the flux through B:

$$\iint_B \vec{F}\bullet d\vec{S} = \int_0^{2\pi}\int_0^2 3r^3\,dr\,d\theta = 24\pi$$

v. Compute the right hand side:

$$\iint_{\partial V} \vec{F}\bullet d\vec{S} = \iint_C \vec{F}\bullet d\vec{S} + \iint_T \vec{F}\bullet d\vec{S} + \iint_B \vec{F}\bullet d\vec{S} = 144\pi + 24\pi + 24\pi = 192\pi$$ which agrees with (i).

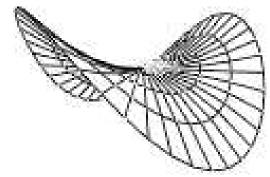
12. (15 points) Compute $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ for $\vec{F} = (-y, x, z)$ over the hyperbolic paraboloid $z = x^2 - y^2$ parametrized by

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2(\cos^2\theta - \sin^2\theta))$$

for $r \le 2$ oriented upward.

HINTS: Use Stokes Theorem.

What is the value of r on the boundary?



SOLUTION:

Boundary: $r = 2$ $\quad \vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 4(\cos^2\theta - \sin^2\theta))$ $\quad \vec{v} = (-2\sin\theta, 2\cos\theta, -16\cos\theta\sin\theta)$

$\vec{F}(\vec{r}(\theta)) = (-2\sin\theta, 2\cos\theta, 4(\cos^2\theta - \sin^2\theta))$

$\vec{F} \cdot \vec{v} = 4\sin^2\theta + 4\cos^2\theta - 64(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta) = 4 - 64(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)$

$$\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 4 - 64(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta)\, d\theta$$

$$= \Big[4\theta - 16(-\cos^4\theta - \sin^4\theta) \Big]_0^{2\pi} = 8\pi$$