Name________________________

MATH 221 Exam 1 Version A Fall 2019

Section 504 Solutions P. Yasskin

1-9	/54	11	/15
10	/36	Total	/105

Multiple Choice: (6 points each. No part credit.)

1. Find the angle between the vectors $\vec{a} = \langle 1,2,1 \rangle$ and $\vec{b} = \langle 0,1,1 \rangle$.

a. $0°$

b. $30°$ Correct Choice

c. $45°$

d. $60°$

e. $90°$

Solution: $\vec{a} \cdot \vec{b} = 2 + 1 = 3$ $\quad |\vec{a}| = \sqrt{1+4+1} = \sqrt{6}$ $\quad \left|\vec{b}\right| = \sqrt{1+1} = \sqrt{2}$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\left|\vec{b}\right|} = \frac{3}{\sqrt{6}\,\sqrt{2}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2} \qquad \theta = 30°$$

2. Two tugboats are pulling on a barge with the forces:

$\vec{F}_1 = \langle 4,2 \rangle$ and $\vec{F}_2 = \langle -2,1 \rangle$

They move the barge from $P = (1,0)$ to $Q = (2,4)$. Find the work done.

a. 20

b. 18

c. 16

d. 14 Correct Choice

e. 12

Solution: The force is $\vec{F} = \vec{F}_1 + \vec{F}_2 = \langle 4,2 \rangle + \langle -2,1 \rangle = \langle 2,3 \rangle$. The displacement is $\vec{D} = \overrightarrow{PQ} = Q - P = (2,4) - (1,0) = (1,4)$. So the work is $W = \vec{F} \cdot \vec{D} = 2 + 12 = 14$.

3. If $\vec{u}$ points West and $\vec{v}$ points NorthEast, where does $\vec{u} \times \vec{v}$ point?

a. Down Correct Choice

b. Up

c. SouthWest

d. SouthEast

e. South

Solution: Hold your right hand with the fingers pointing West and the palm facing NorthEast. Then the thumb points Down.

4. If $|\vec{u}| = 2$, $|\vec{v}| = 5$ and $\vec{u} \cdot \vec{v} = 6$, what is $|\vec{u} \times \vec{v}|$?

a. 64

b. 8 Correct Choice

c. 6

d. 4

e. 2

Solution: $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2|\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 = 2^2 5^2 - 6^2 = 100 - 36 = 64$ So $|\vec{u} \times \vec{v}| = 8$.

5. Find the area of the triangle with vertices $A = (2,3,4)$, $B = (4,3,2)$ and $C = (4,2,4)$.

a. 12

b. $\sqrt{12}$

c. 6

d. $\sqrt{6}$ Correct Choice

e. $\sqrt{3}$

Solution: Two edges are $\overrightarrow{AB} = B - A = \langle 2,0,-2 \rangle$ and $\overrightarrow{AC} = C - A = \langle 2,-1,0 \rangle$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 0 & -2 \\ 2 & -1 & 0 \end{vmatrix} = \hat{\imath}(0-2) - \hat{\jmath}(0 - -4) + \hat{k}(-2-0) = \langle -2,-4,-2 \rangle$$

$$A = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{4+16+4} = \frac{1}{2}\sqrt{24} = \sqrt{6}$$

6. Find a vector $\vec{w}$ of length 6 in the same direction as $\vec{v} = \langle 2,1,-2 \rangle$. The sum of its components is

a. 1

b. 2 Correct Choice

c. 6

d. 8

e. 12

Solution: $|\vec{v}| = \sqrt{4+1+4} = 3$ $\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{-2}{3} \right\rangle$

We want $|\vec{w}| = 6$ and $\hat{w} = \hat{v} = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{-2}{3} \right\rangle$. So $\vec{w} = |\vec{w}|\hat{w} = 6\left\langle \frac{2}{3}, \frac{1}{3}, \frac{-2}{3} \right\rangle = \langle 4,2,-4 \rangle$.

The sum of its components is $4 + 2 - 4 = 2$.

7. Classify the surface: $2x^2 - 8x - y^2 + 6y + z^2 = 2$.

a. Hyperbolic Paraboloid

b. Hyperbolic Cylinder

c. Hyperboloid of 1 sheet   Correct Choice

d. Hyperboloid of 2 sheets

e. Cone

Solution: Complete the squares: $2(x^2 - 2x + 4) - (y^2 - 6x + 9) + z^2 = 2 + 8 - 9$
$2(x-2)^2 - (y-3)^2 + z^2 = 1$ Hyperboloid of 1 sheet

8. Find the point where the line $(x,y,z) = (1+3t, 2+2t, 3+t)$ intersects the plane $2x - y + z = 13$.
The sum of the components is:

a. -6

b. 6

c. 12

d. 18 Correct Choice

e. No intersection. They are parallel.

Solution: Substitute the line into the plane and solve for t:
$2(1+3t) - (2+2t) + 3 + t = 13$ or $3 + 5t = 13$ or $t = 2$
Substitute back into the line: $(x,y,z) = (1 + 3 \cdot 2, 2 + 2 \cdot 2, 3 + 2) = (7,6,5)$
Check in the plane: $2 \cdot 7 - 6 + 5 = 13$
The sum of the components is: $7 + 6 + 5 = 18$

9. Find the plane through the point $P = (0,5,3)$ with tangent vectors $\vec{u} = \langle 2,1,3 \rangle$ and $\vec{v} = \langle -1,2,-2 \rangle$.
Its z-intercept is:

a. $z = 5$

b. $z = 10$

c. $z = 20$

d. $z = 2$

e. $z = 4$ Correct Choice

Solution: The normal is

$$\vec{N} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 3 \\ -1 & 2 & -2 \end{vmatrix} = \hat{\imath}(-2-6) - \hat{\jmath}(-4+3) + \hat{k}(4+1) = \langle -8,1,5 \rangle$$

The plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or $-8x + y + 5z = -8(0) + (5) + 5(3) = 20$.
The z-intercept satisfies $5z = 20$ or $z = 4$.

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (36 points) For the curve $\vec{r}(t) = \langle t, 2e^t, e^{2t} \rangle$ compute each of the following:

a. (6 pts) The velocity $\vec{v}$

Solution: $\vec{v} = \underline{\langle 1, 2e^t, 2e^{2t} \rangle}$

b. (6 pts) The speed $\frac{ds}{dt}$ (Simplify!)

Solution: $\frac{ds}{dt} = |\vec{v}| = \sqrt{1 + 4e^{2t} + 4e^{4t}} = \sqrt{(1 + 2e^{2t})^2} = 1 + 2e^{2t}$

$\frac{ds}{dt} = \underline{1 + 2e^{2t}}$

c. (6 pts) The tangential acceleration a_T

Solution: $a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}(1 + 2e^{2t}) = 4e^{2t}$

$a_T = \underline{4e^{2t}}$

d. (6 pts) The length of this curve between $(0,2,1)$ and $(1,2e,e^2)$.

Solution: $|\vec{v}| = 1 + 2e^{2t}$ $\quad (0,2,1) = \vec{r}(0) \quad (1,2e,e^2) = \vec{r}(1)$

$$L = \int_{(0,2,1)}^{(1,2e,e^2)} ds = \int_0^1 |\vec{v}|\, dt = \int_0^1 (1 + 2e^{2t})\, dt = \Big[t + e^{2t}\Big]_0^1 = 1 + e^2 - 1 = e^2$$

$L = \underline{e^2}$

e. (6 pts) The mass of a wire in the shape of this curve between $(0,2,1)$ and $(1,2e,e^2)$ if the linear mass density is $\delta = yz$.

Solution: $|\vec{v}| = 1 + 2e^{2t}$ $\quad \delta = yz = 2e^t e^{2t} = 2e^{3t}$ $\quad M = \int_{(0,2,1)}^{(1,2e,e^2)} \delta\, ds = \int_0^1 yz|\vec{v}|\, dt$

$$M = \int_0^1 2e^{3t}(1 + 2e^{2t})\, dt = \left[\frac{2e^{3t}}{3} + \frac{4e^{5t}}{5}\right]_0^1 = \frac{2e^3}{3} + \frac{4e^5}{5} - \frac{2}{3} - \frac{4}{5}$$

$M = \underline{\frac{2e^3}{3} + \frac{4e^5}{5} - \frac{2}{3} - \frac{4}{5}}$

f. (6 pts) The work done to move a bead along of a wire in the shape of this curve between $(0,2,1)$ and $(1,2e,e^2)$ by the force $\vec{F} = \langle 0, z, y \rangle$.

Solution: $\vec{F}(\vec{r}(t)) = \langle 0, z, y \rangle = \langle 0, e^{2t}, 2e^t \rangle$ $\quad \vec{v} = \langle 1, 2e^t, 2e^{2t} \rangle$

$$\vec{F} \cdot \vec{v} = e^{2t} \cdot 2e^t + 2e^t \cdot 2e^{2t} = 6e^{3t}$$

$$W = \int_{(0,2,1)}^{(1,2e,e^2)} \vec{F} \cdot d\vec{s} = \int_0^1 \vec{F} \cdot \vec{v}\, dt = \int_0^1 6e^{3t}\, dt = \Big[2e^{3t}\Big]_0^1 = 2e^3 - 2$$

$W = \underline{2e^3 - 2}$

11. (15 points) Consider the two straight lines:

$$L_1 : \qquad (x,y,z) = (2+t,3,4+2t)$$
$$L_2 : \qquad (x,y,z) = (1,2+t,3-2t)$$

Are they parallel or skew or do they intersect? If they intersect, find the point of intersection

Solution: The direction vectors are $\vec{v}_1 = \langle 1,0,2 \rangle$ and $\vec{v}_2 = \langle 0,1,-2 \rangle$. Since one is not a multiple of the other, the lines are not parallel. We first change the parameter name on the second line:

$$L_2 : \qquad (x,y,z) = (1,2+s,3-2s)$$

We equate the x, y and z components:

$$2+t = 1$$
$$3 = 2+s$$
$$4+2t = 3-2s$$

The first two equations say $t=-1$ and $s=1$. Using these, the third equation says

$$4+2(-1) = 3-2(1) \qquad \text{or} \qquad 2 = 1$$

This is impossible. So there is no solution. There is no intersection. The lines are skew!