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MATH 221 Exam 3 Version A Fall 2019

Section 505 Solutions P. Yasskin

1-9	/54	11	/25
10	/25	Total	/104

Multiple Choice: (6 points each. No part credit.)

1. Compute the integral $\iint x^2 y\,dA$ over the region between $y = 4x^2$ and $y = x^4$ in the first quadrant.

a. $\dfrac{2^{12}}{77}$ Correct Choice

b. $\dfrac{2^{13}}{77}$

c. $\dfrac{2^6}{35}$

d. $\dfrac{2^7}{35}$

e. $\dfrac{2^8}{35}$

Solution: The curves intersect when $4x^2 = x^4$ or $x = 0, \pm 2$. At $x = 1$, $4x^2 > x^4$. So

$$\iint x^2 y\,dA = \int_0^2 \int_{x^4}^{4x^2} x^2 y\,dy\,dx = \int_0^2 \left[x^2 \frac{y^2}{2} \right]_{y=x^4}^{4x^2} dx = \int_0^2 \left(x^2 \frac{16x^4}{2} \right) - \left(x^2 \frac{x^8}{2} \right) dx$$

$$= \left[8\frac{x^7}{7} - \frac{1}{2}\frac{x^{11}}{11} \right]_0^2 = \frac{2^{10}}{7} - \frac{2^{10}}{11} = \frac{2^{12}}{77}$$

2. Find the average value of the function $f(x,y) = x^2 y$ on the rectangle $[0,3] \times [0,4]$.

a. $\dfrac{3}{2}$

b. 6 Correct Choice

c. 12

d. 18

e. 72

Solution: The area is $A = 3 \cdot 4 = 12$.

$$\int_0^4 \int_0^3 x^2 y\,dx\,dy = \left[\frac{x^3}{3} \right]_0^3 \left[\frac{y^2}{2} \right]_0^4 = 9 \cdot 8 = 72 \qquad f_{\text{ave}} = \frac{1}{A} \int_0^4 \int_0^3 x^2 y\,dx\,dy = \frac{72}{12} = 6$$

3. Compute $\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy$

a. $\frac{e^8}{3} - \frac{1}{3}$ Correct Choice

b. $\frac{e^{16}}{4} - \frac{1}{4}$

c. $\frac{e^{16}}{4} - \frac{e^4}{4}$

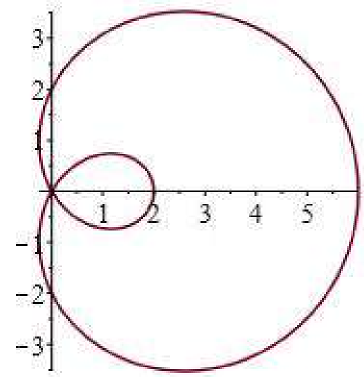
d. $\frac{e^{64}}{4} - \frac{1}{4}$

e. $\frac{e^{64}}{4} - \frac{e^{16}}{4}$

Solution: We reverse the order of integration.

$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\,dx\,dy = \int_0^2 \int_0^{x^2} e^{x^3}\,dy\,dx = \int_0^2 x^2 e^{x^3}\,dx = \left[\frac{1}{3}e^{x^3}\right]_0^2 = \frac{e^8}{3} - \frac{1}{3}$$

4. Find the area of the **inner loop** of the limacon $r = 4\cos\theta - 2$.
HINT: Find the angles at which $r = 0$.



a. $2\pi + 6\sqrt{3}$

b. $4\pi - 6\sqrt{3}$ Correct Choice

c. $6\pi - 4\sqrt{3}$

d. $8\pi + 4\sqrt{3}$

e. $8\pi + 6\sqrt{3}$

Solution: The loop passes thru the origin when $r = 0$. So

$$4\cos\theta - 2 = 0 \qquad \cos\theta = \frac{1}{2} \qquad \theta = \pm\frac{\pi}{3}$$

$$A = \int_{-\pi/3}^{\pi/3} \int_0^{4\cos\theta - 2} r\,dr\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (4\cos\theta - 2)^2\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (16\cos^2\theta - 16\cos\theta + 4)\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (16\cos^2\theta - 16\cos\theta + 4)\,d\theta = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (8(1 + \cos 2\theta) - 16\cos\theta + 4)\,d\theta$$

$$= \frac{1}{2}[12\theta + 4\sin 2\theta - 16\sin\theta]_{-\pi/3}^{\pi/3} = 12\frac{\pi}{3} + 4\sin\frac{2\pi}{3} - 16\sin\frac{\pi}{3} = 4\pi - 6\sqrt{3}$$

5. Find the mass of the solid between the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$ if the density is $\delta = z$.

a. 4π

b. 32π

c. 36π

d. 64π Correct Choice

e. 120π

Solution: In cylindrical coordinates, the parabolas are $z = r^2$ and $z = 8 - r^2$. They intersect when

$$r^2 = 8 - r^2 \qquad r^2 = 4 \qquad r = 2$$

$$M = \iiint \delta\, dV = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} zr\, dz\, dr\, d\theta = 2\pi\int_0^2 \left[\frac{z^2}{2}\right]_{z=r^2}^{8-r^2} r\, dr = \pi\int_0^2 \left[(8-r^2)^2 - r^4\right] r\, dr$$

$$= \pi\int_0^2 [64 - 16r^2] r\, dr = \pi[32r^2 - 4r^4]_0^2 = \pi(128 - 64) = 64\pi$$

6. Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ below the sphere $x^2 + y^2 + z^2 = 9$.

a. 9π

b. 18π

c. $9\pi\sqrt{2}$

d. $9\pi\left(1 - \frac{\sqrt{3}}{2}\right)$

e. $18\pi\left(1 - \frac{1}{\sqrt{2}}\right)$ Correct Choice

Solution: In spherical coordinates, the sphere is $\rho = 3$ and the cone is $\rho\cos\varphi = \rho\sin\varphi$ or $\varphi = \frac{\pi}{4}$.

$$V = \iiint 1\, dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^3 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\left[\frac{\rho^3}{3}\right]_0^3 \left[-\cos\varphi\right]_0^{\pi/4} = 18\pi\left(1 - \frac{1}{\sqrt{2}}\right)$$

7. Compute the circulation of the vector field $\vec{F} = \langle -yz, xz, z^2\rangle$ counterclockwise around the circle $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 2)$. Note: $Circ = \oint \vec{F}\cdot d\vec{s}$

a. 4π

b. 9π

c. 36π Correct Choice

d. 54π

e. 72π

Solution: $\vec{F}(\vec{r}(\theta)) = \langle -6\sin\theta, 6\cos\theta, 4\rangle \qquad \vec{v} = \langle -3\sin\theta, 3\cos\theta, 0\rangle$

$$Circ = \oint \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \vec{F}\cdot\vec{v}\, d\theta = \int_0^{2\pi} (18\sin^2\theta + 18\cos^2\theta)\, d\theta = 36\pi$$

8. Find the x-component of the center of mass of the twisted helix $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ for $0 \le t \le 3$, if the density is $\delta = 2$.

a. $\bar{x} = 42$

b. $\bar{x} = 90$

c. $\bar{x} = \frac{15}{14}$

d. $\bar{x} = \frac{14}{15}$

e. $\bar{x} = \frac{15}{7}$ Correct Choice

Solution: $\vec{v} = (1, 2t, 2t^2)$ $|\vec{v}| = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2$

$M = \int \delta |\vec{v}|\, dt = \int_0^3 2(1 + 2t^2)\, dt = \left[2t + \frac{4}{3}t^3\right]_0^3 = 6 + 36 = 42$

$M_x = \int x \delta |\vec{v}|\, dt = \int_0^3 t\, 2(1 + 2t^2)\, dt = [t^2 + t^4]_0^3 = 9 + 81 = 90$

$\bar{x} = \frac{M_x}{M} = \frac{90}{42} = \frac{15}{7}$

9. Compute $\iiint \vec{\nabla} \cdot \vec{F}\, dV$ over the solid between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 16$ if $\vec{F} = \langle xz^2, yz^2, z^3 \rangle$.

a. $\frac{4^6 \pi}{3}$

b. $\frac{2^7 \pi}{3}$

c. $\frac{4\pi}{3}(4^5 - 2^5)$ Correct Choice

d. $\frac{2\pi}{3}(4^5 - 2^5)$

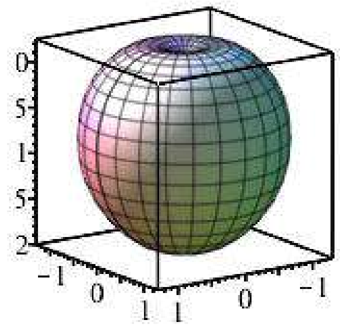
e. $\frac{2\pi}{3}(4^6 - 2^6)$

Solution: $\vec{\nabla} \cdot \vec{F} = z^2 + z^2 + 3z^2 = 5z^2 = 5\rho^2 \cos^2\varphi$

$\iiint \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^{\pi} \int_2^4 5\rho^2 \cos^2\varphi\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$ Let $u = \cos\varphi$ and $du = -\sin\varphi\, d\varphi$.

$\iiint \vec{\nabla} \cdot \vec{F}\, dV = -2\pi \left[\frac{5\rho^5}{5}\right]_2^4 \int_1^{-1} u^2\, du = -2\pi(4^5 - 2^5)\left[\frac{u^3}{3}\right]_1^{-1} = \frac{4\pi}{3}(4^5 - 2^5)$

10. (25 points) The apple at the right is given in spherical coordinates by $\rho = 1 - \cos\varphi$.



HINTS: $(1-u)^3 = 1 - 3u + 3u^2 - u^3$

$(1-u)^4 = 1 - 4u + 6u^2 - 4u^3 + u^4$

a. Find the volume.

Solution:

$$V = \iiint 1\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi \int_0^{\pi} \sin\varphi \left[\frac{\rho^3}{3}\right]_0^{1-\cos\varphi} d\varphi = \frac{2\pi}{3}\int_0^{\pi}(1-\cos\varphi)^3 \sin\varphi\, d\rho$$

Let $u = \cos\varphi$. So $du = -\sin\varphi\, d\varphi$. Then

$$V = \frac{-2\pi}{3}\int_1^{-1}(1-u)^3\, du = \frac{2\pi}{3}\int_{-1}^{1}(1 - 3u + 3u^2 - u^3)\, du = \frac{2\pi}{3}\left[u - \frac{3u^2}{2} + u^3 - \frac{u^4}{4}\right]_{-1}^{1}$$

Even powers cancel. Odd powers add up.

$$V = \frac{4\pi}{3}(1 + 1^3) = \frac{8\pi}{3}$$

b. Find the centroid. (Part credit: Set up the integrals and say what you do with the answers.)

Solution: $\bar{x} = \bar{y} = 0$ by symmetry.

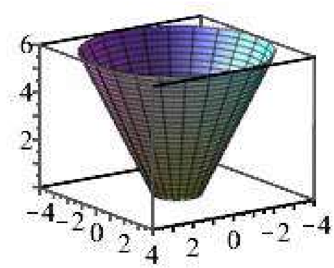
$$V_z = \iiint z\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\varphi} \rho\cos\varphi\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\int_0^{\pi}\cos\varphi\sin\varphi\left[\frac{\rho^4}{4}\right]_0^{1-\cos\varphi} d\varphi$$

$$= \frac{\pi}{2}\int_0^{\pi}\cos\varphi(1-\cos\varphi)^4\sin\varphi\, d\rho\, d\varphi = \frac{-\pi}{2}\int_1^{-1} u(1-u)^4\, du = \frac{\pi}{2}\int_{-1}^{1} u(1 - 4u + 6u^2 - 4u^3 + u^4)\, du$$

$$= \frac{\pi}{2}\left[\frac{u^2}{2} - \frac{4u^3}{3} + \frac{6u^4}{4} - \frac{4u^5}{5} + \frac{u^6}{6}\right]_{-1}^{1} = \pi\left(-\frac{4}{3} - \frac{4}{5}\right) = -\frac{32}{15}\pi$$

$$\bar{z} = \frac{V_z}{V} = -\frac{32\pi}{15}\frac{3}{8\pi} = -\frac{4}{5}$$

11. (25 points) Consider a bowl given in cylindrical coordinates by $z = 2r - 2$ for $1 \le r \le 4$ oriented **down and out**. The surface may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 2r-2)$.



a. Find the surface area.

i. Find the tangent vectors:

$$\vec{e}_r = (\cos\theta, \sin\theta, 2)$$
$$\vec{e}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

ii. Find the normal:

$$\vec{N} = \hat{\imath}(-2r\cos\theta) - \hat{\jmath}(2r\sin\theta) + \hat{k}(r)$$

iii. Find the length of the normal:

$$\left|\vec{N}\right| = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2} = \sqrt{4r^2 + r^2} = \sqrt{5}\,r$$

iv. Find the area:

$$A = \iint 1\,dS = \iint 1\left|\vec{N}\right| dr\,d\theta = \int_0^{2\pi}\int_1^4 \sqrt{5}\,r\,dr\,d\theta = 2\pi\sqrt{5}\left[\frac{r^2}{2}\right]_1^4 = \pi\sqrt{5}\,(16-1) = 15\pi\sqrt{5}$$

b. Find the flux of the vector field $\vec{F} = \langle x, y, -z\rangle$ **down** through the surface.

i. Evaluate the vector field on the surface:

$$\vec{F}\Big|_{\vec{R}} = \langle r\cos\theta, r\sin\theta, -2r+2\rangle$$

ii. Restate the normal:

$\vec{N} = (-2r\cos\theta, -2r\sin\theta, r)$ This is up and in.

Reverse: $\vec{N} = (2r\cos\theta, 2r\sin\theta, -r)$ This is down and out.

iii. Compute the flux:

$$Flux = \iint \vec{F}\cdot d\vec{S} = \iint \vec{F}\Big|_{\vec{R}} \cdot \vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_1^4 (2r^2\cos^2\theta + 2r^2\sin^2\theta + (2r-2)r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^4 (4r^2 - 2r)\,dr\,d\theta = 2\pi\left[\frac{4r^3}{3} - r^2\right]_1^4 = 2\pi\left(\frac{256}{3} - 16\right) - 2\pi\left(\frac{4}{3} - 1\right)$$

$$= 2\pi\left(\frac{252}{3} - 15\right) = 2\pi(84 - 15) = 138\pi$$