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MATH 221 Final Exam Version B Fall 2019

Section 505 Solutions P. Yasskin

1-11	/55	13	/10
12	/15	14	/25
		Total	/105

Multiple Choice: (5 points each. No part credit.)

1. A triangle has vertices $A = (3,2,1)$, $B = (3,3,2)$ and $C = (4,4,2)$. Find the angle at A.

a. $0°$

b. $30°$ Correct Choice

c. $45°$

d. $60°$

e. $90°$

Solution: $\overrightarrow{AB} = B - A = (0,1,1)$ $\overrightarrow{AC} = C - A = (1,2,1)$ $\left|\overrightarrow{AB}\right| = \sqrt{2}$ $\left|\overrightarrow{AC}\right| = \sqrt{6}$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 0 + 2 + 1 = 3$ $\cos\theta = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \dfrac{3}{\sqrt{2}\sqrt{6}} = \dfrac{\sqrt{3}}{2}$ $\theta = 30°$

2. Find the area of the triangle with vertices $A = (3,2,1)$, $B = (3,3,2)$ and $C = (4,4,2)$.

a. $2\sqrt{6}$

b. $\sqrt{6}$

c. $\dfrac{\sqrt{6}}{2}$

d. $\sqrt{3}$

e. $\dfrac{\sqrt{3}}{2}$ Correct Choice

Solution: $\overrightarrow{AB} = B - A = \langle 0,1,1 \rangle$ $\overrightarrow{AC} = C - A = \langle 1,2,1 \rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \hat{\imath}(1-2) - \hat{\jmath}(0-1) + \hat{k}(0-1) = \langle -1,1,-1 \rangle \qquad A = \frac{\sqrt{3}}{2}$$

3. Find the arc length of the curve $\vec{r}(t) = (t^2, 2t, \ln t)$ between $(1,2,0)$ and $(e^2, 2e, 1)$.

a. $e^2 + 1$
b. $e^2 - 1$
c. e^2 Correct Choice
d. $\sqrt{e^2+1}$
e. $\sqrt{e^2+1} - 1$

Solution: $|\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(2t)^2 + (2)^2 + (t^{-1})^2} = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = 2t + \frac{1}{t}$

$L = \int_1^e ds = \int_1^e |\vec{v}|dt = \int_1^e \left(2t + \frac{1}{t}\right)dt = \left[t^2 + \ln t\right]_1^e = (e^2+1) - (1-0) = e^2$

4. Find the average value of the function $f(x,y,z) = x$ along the curve $\vec{r}(t) = (t^2, 2t, \ln t)$ between $(1,2,0)$ and $(e^2, 2e, 1)$.

a. $\frac{e^2}{2} - \frac{1}{e^2}$
b. $\frac{e^4}{2} - \frac{e^2}{2} - 1$
c. $\frac{e^2}{2} - \frac{1}{2} - \frac{1}{e^2}$
d. $\frac{e^4}{2} + \frac{e^2}{2} - 1$
e. $\frac{e^2}{2} + \frac{1}{2} - \frac{1}{e^2}$ Correct Choice

Solution: $f_{\text{ave}} = \frac{1}{L}\int_1^e f ds = \frac{1}{e^2}\int_1^e x|\vec{v}|dt = \frac{1}{e^2}\int_1^e t^2\left(2t + \frac{1}{t}\right)dt = \frac{1}{e^2}\int_1^e (2t^3 + t)dt = \frac{1}{e^2}\left[\frac{t^4}{2} + \frac{t^2}{2}\right]_1^e$
$= \left(\frac{e^2}{2} + \frac{1}{2}\right) - \frac{1}{e^2}\left(\frac{1}{2} + \frac{1}{2}\right) = \frac{e^2}{2} + \frac{1}{2} - \frac{1}{e^2}$

5. Find the plane tangent to $x\sin y + 2z\cos y = 2$ at $(x,y,z) = \left(\sqrt{2}, \frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$. The z-intercept is:

a. 2
b. $\sqrt{2}$ Correct Choice
c. $\frac{1}{\sqrt{2}}$
d. $\frac{1}{2\sqrt{2}}$
e. 0

Solution: Let $f = x\sin y + 2z\cos y$ and $P = \left(\sqrt{2}, \frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$. Then
$\vec{\nabla}f = \langle \sin y, x\cos y - 2z\sin y, 2\cos y\rangle$ and
$\vec{N} = \vec{\nabla}f\big|_P = \left\langle \frac{1}{\sqrt{2}}, \sqrt{2}\frac{1}{\sqrt{2}} - 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}, 2\frac{1}{\sqrt{2}}\right\rangle = \left\langle \frac{1}{\sqrt{2}}, 0, \sqrt{2}\right\rangle$.
So the tangent plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or $\frac{1}{\sqrt{2}}x + \sqrt{2}z = \frac{1}{\sqrt{2}}\sqrt{2} + \sqrt{2}\frac{1}{\sqrt{2}} = 2$.
The z-intercept occurs when $x = y = 0$ or $\sqrt{2}z = 2$ or $z = \sqrt{2}$.

6. The velocity field of the water in a sink is $\vec{V} = \langle -x^2 y, xy^2 \rangle$ Find the circulation of the water, $Circ = \oint \vec{V} \cdot d\vec{s}$, counterclockwise around the circle $x^2 + y^2 = 4$.

HINT: Use Green's Theorem.

a. 2π
b. $\frac{16}{3}\pi$
c. 8π Correct Choice
d. 32π
e. 64π

Solution: $Circ = \oint \vec{V} \cdot d\vec{s} = \oint P\,dx + Q\,dy$ where $P = -x^2 y$ and $Q = xy^2$.
By Green's Theorem, $Circ = \iint (\partial_x Q - \partial_y P)\,dx\,dy = \iint (y^2 - -x^2)\,dx\,dy = \iint (x^2 + y^2)\,dx\,dy$
In polar coordinates, $Circ = \int_0^{2\pi} \int_0^2 r^2 r\,dr\,d\theta = 2\pi \left[\frac{r^4}{4} \right]_0^2 = 8\pi$

7. Compute $\int \vec{F} \cdot d\vec{s}$ for $\vec{F} = \langle 4x^3, 3y^2, 2z \rangle$ along the curve $\vec{r}(t) = \left(t\sin\left(\frac{\pi}{2}t\right), 2^t \cos\left(\frac{\pi}{2}t\right), 2^t \sin\left(\frac{\pi}{2}t\right) \right)$ from $t = 0$ to $t = 1$.

HINT: Find a scalar potential.

a. 0
b. 1
c. 2
d. 3
e. 4 Correct Choice

Solution: A scalar potential is $f = x^4 + y^3 + z^2$.
The endpoints are $A = \vec{r}(0) = (0,1,0)$ and $B = \vec{r}(1) = (1,0,2)$.
By the FTCC, $\int \vec{F} \cdot d\vec{s} = \int_A^B \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = (1^4 + 2^2) - (1^3) = 4$

8. Compute $\iint_S \vec{\nabla} \times \vec{G} \cdot d\vec{S}$ for $\vec{G} = \langle y, -x, z \rangle$ over the surface $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 16 - r^4)$ for $0 \le r \le 2$ and $0 \le \theta \le 2\pi$ oriented up and out.

HINT: Use a theorem.

a. -8π Correct Choice
b. -4π
c. -2π
d. 2π
e. 4π

Solution: The boundary curve $(r = 2)$ is $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 0)$.
This correctly points counterclockwise. The velocity is $\vec{v} = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$.
On the circle, $\vec{G} = \langle y, -x, z \rangle = \langle 2\sin\theta, -2\cos\theta, 0 \rangle$. By Stokes' Theorem:

$$\iint_S \vec{\nabla} \times \vec{G} \cdot d\vec{S} = \oint_{\partial S} \vec{G} \cdot d\vec{s} = \int_0^{2\pi} \vec{G} \cdot \vec{v}\,d\theta = \int_0^{2\pi} -4\sin^2\theta - 4\cos^2\theta\,d\theta = -8\pi$$

9. Find the centroid of the solid hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$.

a. $\left(0, 0, \frac{1}{2}\right)$

b. $\left(0, 0, \frac{3}{4}\right)$ Correct Choice

c. $\left(0, 0, \frac{4}{3}\right)$

d. $(0, 0, 4\pi)$

e. $(0, 0, 2\pi)$

Solution: By symmetry $\bar{x} = \bar{y} = 0$.
In spherical coordinates, $z = \rho\cos\varphi$ and $dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$.

$$V_z = \iiint z \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho\cos\varphi \, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = 2\pi \left[\frac{\sin^2\varphi}{2} \right]_0^{\pi/2} \left[\frac{\rho^4}{4} \right]_0^2 = 4\pi$$

$$V = \frac{1}{2}\frac{4}{3}\pi 2^3 = \frac{16}{3}\pi \qquad \bar{z} = \frac{V_z}{V} = \frac{3 \cdot 4\pi}{16\pi} = \frac{3}{4}$$

10. Find the area of the surface $\vec{R}(u,v) = (u, v, u + v)$ for $0 \le u \le 2$ and $0 \le v \le 3$.

a. $12\sqrt{2}$

b. $12\sqrt{3}$

c. $6\sqrt{2}$

d. $6\sqrt{3}$ Correct Choice

e. $3\sqrt{2}$

Solution: $\vec{e}_u = \langle 1, 0, 1 \rangle$ $\quad \vec{e}_v = \langle 0, 1, 1 \rangle$ $\quad \vec{N} = \vec{e}_u \times \vec{e}_v = \langle -1, -1, 1 \rangle$ $\quad \left|\vec{N}\right| = \sqrt{3}$

$$A = \iint \left|\vec{N}\right| du \, dv = \int_0^3 \int_0^2 \sqrt{3} \, du \, dv = 6\sqrt{3}$$

11. Find the flux of $\vec{F} = \langle y, -z, x \rangle$ upward thru the surface $\vec{R}(u,v) = (u, v, u + v)$ for $0 \le u \le 2$ and $0 \le v \le 3$.

a. 30

b. 24

c. 18

d. 12 Correct Choice

e. 0

Solution: $\vec{N} = \vec{e}_u \times \vec{e}_v = \langle -1, -1, 1 \rangle$ which is correctly upward. $\vec{F} = \langle y, -z, x \rangle = \langle v, -u - v, u \rangle$

$$\iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{N} \, du \, dv = \int_0^3 \int_0^2 (-v - (-u - v) + u) \, du \, dv = \int_0^3 \int_0^2 2u \, du \, dv = [u^2]_0^2 [v]_0^3 = 12$$

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (15 points) The Ideal Gas Law says the Pressure, P, Density, δ, and Temperature, T, are related by $P = k\delta T$ for some constant k. We will assume the atmosphere is an ideal gas with $k = 2$. A weather baloon measures the Density and Temperature to be

$$\delta = 0.1\ \frac{\text{gm}}{\text{m}^3} \qquad T = 270°\text{K}$$

and their gradients to be

$$\vec{\nabla}\delta = \langle 0.01, 0.02, 0.03\rangle \qquad \vec{\nabla}T = \langle 3, 1, -1\rangle$$

Find the gradient of the Pressure.

Solution: $\dfrac{\partial P}{\partial \delta} = kT = 2 \cdot 270 = 540 \qquad \dfrac{\partial P}{\partial T} = k\delta = 2 \cdot 0.1 = 0.2$

By the chain rule,

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial \delta}\frac{\partial \delta}{\partial x} + \frac{\partial P}{\partial T}\frac{\partial T}{\partial x} = 540 \cdot 0.01 + 0.2 \cdot 3 = 6$$

$$\frac{\partial P}{\partial y} = \frac{\partial P}{\partial \delta}\frac{\partial \delta}{\partial y} + \frac{\partial P}{\partial T}\frac{\partial T}{\partial y} = 540 \cdot 0.02 + 0.2 \cdot 1 = 11$$

$$\frac{\partial P}{\partial z} = \frac{\partial P}{\partial \delta}\frac{\partial \delta}{\partial z} + \frac{\partial P}{\partial T}\frac{\partial T}{\partial z} = 540 \cdot 0.03 + 0.2 \cdot (-1) = 16$$

So

$$\vec{\nabla}P = \langle 6, 11, 16\rangle$$

13. (10 points) Find the largest value of $f = xy^2z^3$ on the plane $2x + 3y + 6z = 72$.

Solution 1: Lagrange Multipliers: Let $g = 2x + 3y + 6z$.

Then $\vec{\nabla}f = \langle y^2z^3, 2xyz^3, 3xy^2z^2\rangle$ and $\vec{\nabla}g = \langle 2, 3, 6\rangle$.

The Lagrange equations are: $y^2z^3 = 2\lambda, \quad 2xyz^3 = 3\lambda, \quad 3xy^2z^2 = 6\lambda$

Then $xy^2z^3 = 2\lambda x = \frac{3}{2}\lambda y = 2\lambda z$ or $y = \frac{4}{3}x$ and $z = x$

From the constraint, $72 = 2x + 3y + 6z = 2x + 4x + 6x = 12x$.

So $x = 6$, $y = 8$, $z = 6$ and $f = xy^2z^3 = 6 \cdot 8^2 \cdot 6^3$

Solution 2: Eliminate $x = \dfrac{72 - 3y - 6z}{2} = 36 - \dfrac{3y}{2} - 3z$. Then

$$f = xy^2z^3 = \left(36 - \frac{3y}{2} - 3z\right)y^2z^3 = 36y^2z^3 - \frac{3}{2}y^3z^3 - 3y^2z^4$$

We find critical points:

$$f_y = 72yz^3 - \frac{9}{2}y^2z^3 - 6yz^4 = \frac{3}{2}yz^3(48 - 3y - 4z) = 0$$

$$f_z = 108y^2z^2 - \frac{9}{2}y^3z^2 - 12y^2z^3 = \frac{3}{2}y^2z^2(72 - 3y - 8z) = 0$$

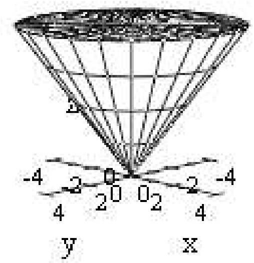
Since $y \neq 0$ and $z \neq 0$, we need to solve $3y + 4z = 48$ and $3y + 8z = 72$

We subtract $4z = 24$ or $z = 6$

Substituting back gives: $3y + 24 = 48 \qquad 3y = 24 \qquad y = 8$

and $x = 36 - \dfrac{3y}{2} - 3z = 36 - 12 - 18 = 6$. So $f = xy^2z^3 = 6 \cdot 8^2 \cdot 6^3$

14. (25 points) Verify Gauss' Theorem $\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_{\partial V} \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = \left(xz, yz, 3z\sqrt{x^2+y^2}\right)$ and the solid cone $\sqrt{x^2+y^2} \le z \le 2$.



Be careful with orientations. Use the following steps:

First the Left Hand Side:

a. Compute the divergence:

$$\vec{\nabla} \cdot \vec{F} = z + z + 3\sqrt{x^2+y^2}$$

b. Express the divergence and the volume element in the appropriate coordinate system:

$$\vec{\nabla} \cdot \vec{F} = 2z + 3r \qquad\qquad dV = r\,dr\,d\theta\,dz$$

c. Compute the left hand side:

$$\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^2 \int_0^{2\pi} \int_r^2 (2z+3r)\, r\,dz\,d\theta\,dr = 2\pi \int_0^2 \Big[z^2 + 3rz\Big]_{z=r}^2 r\,dr$$

$$= 2\pi \int_0^2 \Big[(4+6r) - (r^2+3r^2)\Big] r\,dr = 2\pi \int_0^2 (4r + 6r^2 - 4r^3)\, dr = 2\pi[2r^2 + 2r^3 - r^4]_0^2$$

$$= 2\pi(8 + 16 - 16) = 16\pi$$

Second the Right Hand Side:

The boundary surface consists of a hemisphere H and a disk D with appropriate orientations.

The disk D may be parametrized as: $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 2)$

d. Compute the tangent vectors:

$$\vec{e}_r = (\cos\theta, \quad \sin\theta, \quad 0)$$

$$\vec{e}_\theta = (-r\sin\theta, \quad r\cos\theta, \quad 0)$$

e. Compute the normal vector:

$$\vec{N} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (0,0,r)$$

This is up as required.

f. Evaluate $\vec{F} = \left(xz, yz, 3z\sqrt{x^2+y^2}\right)$ on the disk:

$$\vec{F}\Big|_{\vec{R}(r,\theta)} = (2r\cos\theta, 2r\sin\theta, 6r)$$

g. Compute the dot product:

$$\vec{F} \cdot \vec{N} = 6r^2$$

h. Compute the flux through D:

$$\iint\limits_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^2 6r^2\, dr\, d\theta = 2\pi[2r^3]_0^2 = 32\pi$$

The cone C may be parametrized as: $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

i. Compute the tangent vectors:

$$\vec{e}_r = (\cos\theta, \quad \sin\theta, \quad 1)$$
$$\vec{e}_\theta = (-r\sin\theta, \quad r\cos\theta, \quad 0)$$

j. Compute the normal vector:

$$\vec{N} = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta)$$
$$= (-r\cos\theta, -r\sin\theta, r)$$

This is oriented up and in. We need down and out. Reverse it:

$$\vec{N} = (r\cos\theta, r\sin\theta, -r)$$

k. Evaluate $\vec{F} = \left(xz, yz, 3z\sqrt{x^2+y^2}\right)$ on the cone:

$$\vec{F}\Big|_{\vec{R}(r,\theta)} = (r^2\cos\theta, r^2\sin\theta, 3r^2)$$

l. Compute the dot product:

$$\vec{F}\cdot\vec{N} = r^3\cos^2\theta + r^3\sin^2\theta - 3r^3 = -2r^3$$

m. Compute the flux through C:

$$\iint_C \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^2 -2r^3\,dr\,d\theta = -2\pi\left[\frac{r^4}{2}\right]_0^2 = -16\pi = -16\pi$$

n. Compute the **TOTAL** right hand side:

$$\iint_{\partial V} \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot d\vec{S} + \iint_C \vec{F}\cdot d\vec{S} = 32\pi - 16\pi = 16\pi$$ which agrees with (c).