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1-8	/48	10	/20+5
9	/15	11	/15
		Total	/108

MATH 221 Exam 2 Fall 2021

Sections 504/505 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. Find the equation of the plane tangent to $z = x^2y^4 - \frac{x}{y}$ at $(x,y) = (2,1)$.
Then find the z-intercept.

Solution: $a = 2$ and $b = 1$.

$$f = x^2y^4 - \frac{x}{y} \qquad f(2,1) = 2$$
$$f_x = 2xy^4 - \frac{1}{y} \qquad f_x(2,1) = 3$$
$$f_y = 4x^2y^3 + \frac{x}{y^2} \qquad f_y(2,1) = 18$$

So the tangent plane is

$$\begin{aligned} z &= f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ &= 2 + 3(x-2) + 18(y-1) \\ &= 3x + 18y + 2 - 6 - 18 \\ &= 3x + 18y - 22 \end{aligned}$$

So the z-intercept is $c = \underline{-22}$.

2. Find the plane tangent to the hyperboloid $4x^2 + 9y^2 - 36z^2 = 36$ at the point $(x,y,z) = (3,2,1)$.
Write the plane in the form where the right side is 1.

Solution: $F = 4x^2 + 9y^2 - 36z^2$ $\quad \vec{\nabla}F = \langle 8x, 18y, -72z \rangle$ $\quad \vec{N} = \vec{\nabla}F\big|_{(3,2,1)} = \langle 24, 36, -72 \rangle$

The plane is $\vec{N} \cdot X = \vec{N} \cdot P$ which is

$$\langle 24,36,-72 \rangle \cdot (x,y,z) = \langle 24,36,-72 \rangle \cdot (3,2,1)$$
$$24x + 36y - 72z = 72$$
$$\underline{\frac{1}{3}}x + \underline{\frac{1}{2}}y + \underline{-1}z = 1$$

3. A weather balloon takes measurements at the point $(x,y,z) = (5,8,3)$ km.
It finds the barometric pressure is $P = 1.05$ atm and its gradient is $\vec{\nabla}P = \langle .02, -.03, .04 \rangle$.
Estimate the pressure at $(x,y,z) = (4.7, 7.8, 3.2)$ km.

Solution: We use the linear approximation to estimate the pressure, taking $a = 5$, $b = 8$, $c = 3$:

$$\begin{aligned} P_{\text{tan}}(x,y,z) &= P(5,8,3) + P_x(5,8,3)(x-5) + P_y(5,8,3)(y-8) + P_z(5,8,3)(z-3) \\ &= 1.05 + .02(x-5) - .03(y-8) + .04(z-3) \end{aligned}$$

$$\begin{aligned} P_{\text{tan}}(4.7,7.8,3.2) &= 1.05 + .02(4.7-5) - .03(7.8-8) + .04(3.2-3) = 1.05 + .02(-.3) - .03(-.2) + .04(.2) \\ &= 1.05 - .006 + .006 + .008 = \underline{1.058} \end{aligned}$$

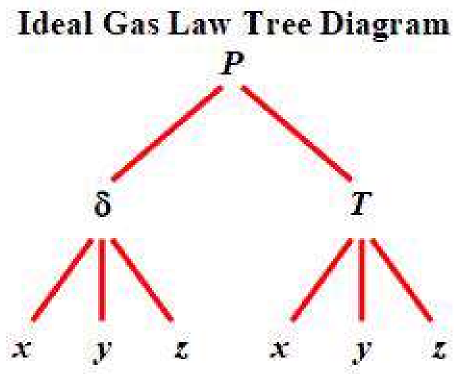
4. The Ideal Gas Law says the Pressure, P, Density, δ, and Temperature, T, are related by $P = k\delta T$.
A particular sample of ideal gas has $k = 2$.
At a certain point the pressure, density and temperature are

$$P = 4 \qquad \delta = .01 \qquad T = 200$$

The gradients of the density and temperature are

$$\vec{\nabla}\delta = \langle .001, .002, .003 \rangle \qquad \vec{\nabla}T = \langle 3, 2, 1 \rangle$$

Find the gradient of the pressure.



Hint: Compute each component separately using the chain rule and the tree diagram at the right.

a. $\vec{\nabla}P = \langle .46, .48, 1.22 \rangle$
b. $\vec{\nabla}P = \langle .64, .48, 1.22 \rangle$
c. $\vec{\nabla}P = \langle .46, .84, 1.22 \rangle$ Correct Choice
d. $\vec{\nabla}P = \langle .64, .84, 1.22 \rangle$
e. $\vec{\nabla}P = \langle .46, .48, 2.11 \rangle$
f. $\vec{\nabla}P = \langle .64, .48, 2.11 \rangle$
g. $\vec{\nabla}P = \langle .46, .84, 2.11 \rangle$
h. $\vec{\nabla}P = \langle .64, .84, 2.11 \rangle$

Solution: The components of the gradients are the 3 partial derivatives.

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial \delta}\frac{\partial \delta}{\partial x} + \frac{\partial P}{\partial T}\frac{\partial T}{\partial x} = 2T\frac{\partial \delta}{\partial x} + 2\delta\frac{\partial T}{\partial x} = 2(200)(.001) + 2(.01)(3) = .4 + .06 = .46$$

$$\frac{\partial P}{\partial y} = \frac{\partial P}{\partial \delta}\frac{\partial \delta}{\partial y} + \frac{\partial P}{\partial T}\frac{\partial T}{\partial y} = 2T\frac{\partial \delta}{\partial y} + 2\delta\frac{\partial T}{\partial y} = 2(200)(.002) + 2(.01)(2) = .8 + .04 = .84$$

$$\frac{\partial P}{\partial z} = \frac{\partial P}{\partial \delta}\frac{\partial \delta}{\partial z} + \frac{\partial P}{\partial T}\frac{\partial T}{\partial z} = 2T\frac{\partial \delta}{\partial z} + 2\delta\frac{\partial T}{\partial z} = 2(200)(.003) + 2(.01)(1) = 1.2 + .02 = 1.22$$

$$\vec{\nabla}P = \langle .46, .84, 1.22 \rangle$$

5. The point $(x,y) = (0,2)$ is a critical point of the function $f(x,y) = y^4 - 32y + 8x^2y$.
Use the Second Derivative Test to classify the point or say the test fails.

a. Local Minimum Correct Choice
b. Local Maximum
c. Inflection Point
d. Saddle Point
e. Test FAILS

Solution: We compute the first derivatives and evaluate at $(x,y) = (0,2)$:

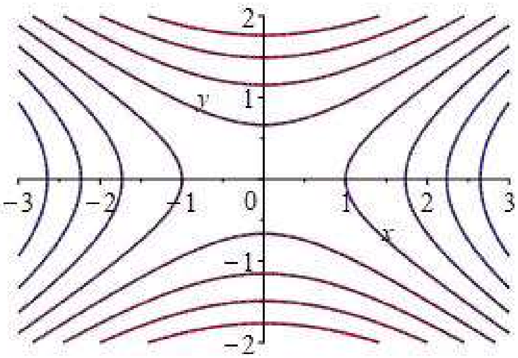
$f_x(x,y) = 16xy$ $\qquad f_x(0,2) = 0$
$f_y(x,y) = 4y^3 - 32 + 8x^2$ $\qquad f_y(0,2) = 4 \cdot 2^3 - 32 = 0$

So we have verified that $(0,2)$ is a critical point.
We compute the second derivatives and evaluate at $(x,y) = (0,2)$:

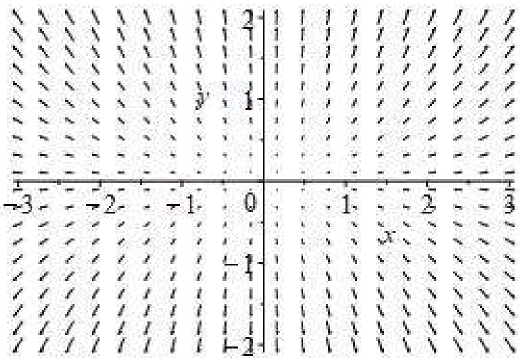
$f_{xx}(x,y) = 16y$ $\qquad f_{xx}(0,2) = 32$
$f_{yy}(x,y) = 12y^2$ $\qquad f_{yy}(0,2) = 48$
$f_{xy}(x,y) = 16x$ $\qquad f_{xy}(0,2) = 0$

So $D(0,2) = f_{xx}f_{yy} - f_{xy}{}^2 = 32 \cdot 48$.
Since $D > 0$ and $f_{xx} > 0$, the point is a local minimum.

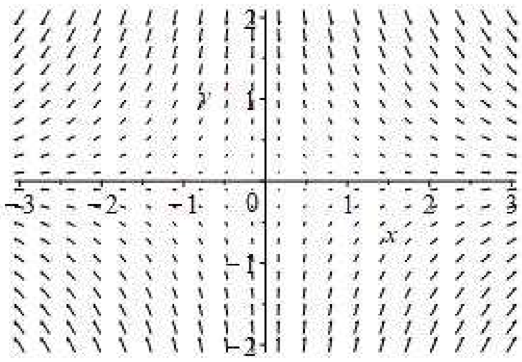
6. The contour plot of a function f is shown. Which of the following is the plot of its gradient, $\vec{\nabla} f$?



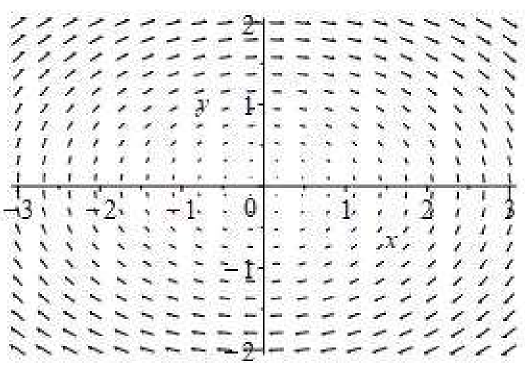
a.



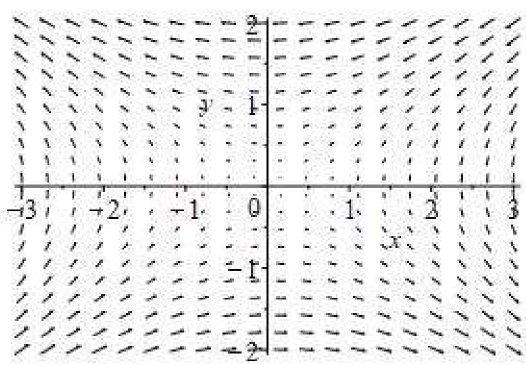
c.



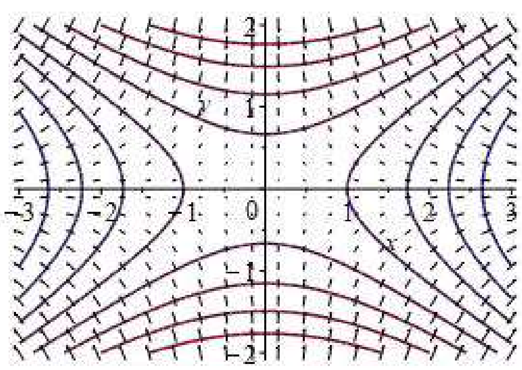
b.



d.



Solution: The answer is (c).
The gradient must be perpendicular to all the level sets.
Here is a plot of (c) superimposed on the contour plot.
You see the vectors are perpendicular to the curves.



7. Ham Duet is flying the Centurion Eagle through a nebula where the density of cloaking sparkles is $\delta = yz + xz + xy$. If Ham's current position is $P = (1,1,3)$, find the rate of change of the density in the direction toward the point $Q = (2,3,1)$.

Solution: Since we want the direction toward Q, we need the directional derivative of δ using a unit vector. The vector from P to Q is $\overrightarrow{PQ} = Q - P = (1,2,-2)$. Its magnitude and direction are

$$\left|\overrightarrow{PQ}\right| = \sqrt{1+4+4} = 3 \qquad \widehat{PQ} = \frac{\overrightarrow{PQ}}{\left|\overrightarrow{PQ}\right|} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{-2}{3} \right\rangle$$

The gradient of the density is $\vec{\nabla}\delta = \langle y+z, x+z, x+y \rangle$. At P this is $\vec{\nabla}\delta\Big|_P = \langle 4,4,2 \rangle$. So the directional derivative is

$$\nabla_{\widehat{PQ}}\delta = \widehat{PQ} \cdot \vec{\nabla}\delta = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{-2}{3} \right\rangle \cdot \langle 4,4,2 \rangle$$
$$= \frac{1}{3}(4+8-4) = \underline{\frac{8}{3}}$$

8. Ham Duet is flying the Centurion Eagle through a nebula where the density of cloaking sparkles is $\delta = yz + xz + xy$. If Ham's current position is $P = (1,1,3)$, in what unit vector direction should he travel to increase the cloaking sparkles as fast as possible?

Solution: The direction of maximum increase is the direction of the gradient.

$$\vec{\nabla}\delta = \langle y+z, x+z, x+y \rangle \qquad \vec{\nabla}\delta\Big|_P = \langle 4,4,2 \rangle \qquad \left|\vec{\nabla}\delta\right| = \sqrt{16+16+2} = 6$$

$$\hat{u} = \frac{\vec{\nabla}\delta}{\left|\vec{\nabla}\delta\right|} = \frac{1}{6}\langle 4,4,2 \rangle = \underline{\left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (15 points) Consider the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$.

Determine the value of the limit or show the limit does not exist.

Solution: $y = mx$: $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{xm^2x^2}{x^2+m^4x^4} = \lim_{(x,y)\to(0,0)} \frac{xm^2}{1+m^4x^2} = 0$

$x = y^2$: $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^2y^2}{y^4+y^4} = \frac{1}{2}$

These are not equal. So the limit does not exist.

10. (20 points + 5 pts extra credit) Find the largest value of $f = xyz$ on the ellipsoid $\frac{x^2}{16} + \frac{y^2}{4} + z^2 = 3$.

NOTE: Solve by either Eliminating a Variable or by Lagrange Multipliers.
Extra Credit for solving by both methods.
Draw a line across your paper to clearly separate the two solutions.
HINT: When Eliminating a Variable, maximize $F = f^2 = x^2y^2z^2$.

Solution 1: Eliminate a Variable Method: We maximize the square of f:

$$F = f^2 = x^2y^2z^2$$

subject to the constraint $z^2 = 3 - \frac{x^2}{16} - \frac{y^2}{4}$. We substitute the constraint into F:

$$F = x^2y^2\left(3 - \frac{x^2}{16} - \frac{y^2}{4}\right) = 3x^2y^2 - \frac{1}{16}x^4y^2 - \frac{1}{4}x^2y^4$$

We set the x and y derivatives equal to 0 and solve for x and y.

$$f_x = 6xy^2 - \frac{1}{4}x^3y^2 - \frac{1}{2}xy^4 = xy^2\left(6 - \frac{1}{4}x^2 - \frac{1}{2}y^2\right) = 0$$

$$f_y = 6x^2y - \frac{1}{8}x^4y - x^2y^3 \quad = x^2y\left(6 - \frac{1}{8}x^2 - y^2\right) = 0$$

Since $x = 0$ or $y = 0$ cannot give a maximum, these are equivalent to:

$$\frac{1}{4}x^2 + \frac{1}{2}y^2 = 6$$

$$\frac{1}{8}x^2 + y^2 = 6$$

We multiply the first equation by 4 and the second equation by 8 and subtract:

$$6y^2 = 24 \qquad \text{So} \quad y = 2$$

Then from the second equation:

$$x^2 = 48 - 8y^2 = 16 \qquad \text{So} \quad x = 4$$

And from the constraint:

$$z^2 = R - \frac{x^2}{16} - \frac{y^2}{4} = 3 - 1 - 1 = 1 \qquad \text{So} \quad z = 1$$

So the function value is $f = xyz = 4 \cdot 2 \cdot 1 = 8$

Solution 2: Lagrange Multiplier Method: We maximize: $f = xyz$

subject to the constraint $g = \frac{x^2}{16} + \frac{y^2}{4} + z^2 = 3$. The gradients are:

$$\vec{\nabla} f = \langle yz, xz, xy \rangle \qquad \vec{\nabla} g = \left\langle \frac{x}{8}, \frac{y}{2}, 2z \right\rangle$$

The Lagrange equations, $\vec{\nabla} f = \lambda \vec{\nabla} g$, are

$$yz = \lambda \frac{x}{8} \qquad xz = \lambda \frac{y}{2} \qquad xy = \lambda 2z$$

We multiply the first equation by x, the second by y and the third by z and equate:

$$xyz = \lambda \frac{x^2}{8} = \lambda \frac{y^2}{2} = \lambda 2z^2$$

So $x^2 = 16z^2$ and $y^2 = 4z^2$. We substitute these into the constraint:

$$z^2 + z^2 + z^2 = 3 \qquad \text{So} \quad z = 1$$

So $x = 4$ and $y = 2$. So the function value is $f = xyz = 4 \cdot 2 \cdot 1 = 8$

11. (15 points) For each vector field, calculate its divergence and curl. Say if it has a scalar potential or a vector potential. Find the scalar potential if it exists. Do NOT find the vector potential.

a. $\vec{F} = \langle -xy^2, x^2y, y^2z - x^2z \rangle$

Solution: $\vec{\nabla} \cdot \vec{F} = -y^2 + x^2 + y^2 - x^2 = 0$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -xy^2 & x^2y & y^2z - x^2z \end{vmatrix} = \hat{\imath}(2yz - 0) - \hat{\jmath}(-2xz - 0) + \hat{k}(2xy - -2xy) = \langle 2yz, 2xz, 2xy \rangle \neq \vec{0}$$

Has a scalar potential? Yes [No] Has a vector potential? [Yes] No

Find a scalar potential: None

b. $\vec{G} = \langle 2xz, 2yz, x^2 + y^2 + 2z \rangle$

Solution: $\vec{\nabla} \cdot \vec{G} = 2z + 2z + 2 \neq 0$

$$\vec{\nabla} \times \vec{G} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2xz & 2yz & x^2 + y^2 + 2z \end{vmatrix} = \hat{\imath}(2y - 2y) - \hat{\jmath}(2x - 2x) + \hat{k}(0 - 0) = \langle 0, 0, 0 \rangle$$

Has a scalar potential: [Yes] No Has a vector potential: Yes [No]

Find a scalar potential:

$$\partial_x g = 2xz \quad \Rightarrow \quad g = x^2z + h(y,z) \quad \Rightarrow \quad \partial_y g = \partial_y h$$

$$\partial_y g = 2yz \quad \Rightarrow \quad \partial_y h = 2yz \quad \Rightarrow \quad h = y^2z + k(z) \quad \Rightarrow \quad \partial_z g = x^2 + y^2 + \frac{dk}{dz}$$

$$\partial_z g = x^2 + y^2 + 2z \quad \Rightarrow \quad \frac{dk}{dz} = 2z \quad \Rightarrow \quad k = z^2$$

$$g = x^2z + y^2z + z^2$$