Name________________________UIN___________________

MATH 221 Final Exam 504 Fall 2021

Sections 504/505 P. Yasskin

1-13	/65	15	/20
14	/15	16	/10
		Total	/110

Multiple Choice: (5 points each. No part credit.)

1. Find the equation of the plane tangent to the graph of $z = f(x,y) = x^3y + xy^2$ at $(2,1)$.
Its z-intercept is

a. -10 f. 10

b. -28 g. 28

c. -35 h. 35

d. -38 i. 38

e. -48 j. 48

2. Find an equation of the plane tangent to the surface $x^2y^2 + x^2z^2 + y^2z^2 = 49$ at the point $(1,2,3)$.

a. $13x + 10y + 5z = 38$ f. $13x - 10y + 5z = 8$

b. $39x + 40y + 30z = 209$ g. $39x - 40y + 30z = 49$

c. $13x + 20y + 15z = 98$ h. $13x - 20y + 15z = 18$

d. $26x + 40y + 30z = 96$ i. $26x - 40y + 30z = 36$

e. $39x + 20y + 5z = 94$ j. $39x - 20y + 5z = 54$

3. You are standing at the airport facing North. You look up and see an airplane circling clockwise above the airport. At the moment when the plane is heading due East, in what direction does the plane's binormal point?

a. North
b. South
c. East
d. West
e. Up
f. Down

4. Find a parametric equation of the line tangent to the curve $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, \theta)$ at the point $(-2, 0, \pi)$.

HINT: What are the point and direction vector?

a. $X(t) = (0, -2 - 2t, \pi + t)$
b. $X(t) = (-2t, -2, 1 + \pi t)$
c. $X(t) = (-2t, 2, 1 + \pi t)$
d. $X(t) = (-2 - 2t, 0, \pi + t)$
e. $X(t) = (-2 - 2t, 0, 1 + \pi t)$
f. $X(t) = (-2, -2t, \pi + t)$
g. $X(t) = (-2, 2t, \pi + t)$
h. $X(t) = (2, -2t, 1 + \pi t)$
i. $X(t) = (2, 2t, \pi + t)$
j. $X(t) = (2, 2t, \pi - t)$

5. Find the arc length of the curve $\vec{r}(t) = (\ln t, 2t, t^2)$ between $(0, 2, 1)$ and $(1, 2e, e^2)$.

Hint: Look for a perfect square.

a. e^2
b. 1
c. $1 + e$
d. $1 + e^2$
e. $e^2 - 1$
f. 2
g. $2 + e$
h. $e - 2$
i. $e^2 - 2$
j. $2 + e^2$

6. For an ideal gas, the pressure, P, is a function of the temperature, T, and volume, V, given by $P = \frac{kT}{V}$ where k is a constant. For a certain sample of gas the current values are

$$T = 250°\text{K} \qquad V = 5\ \text{m}^3 \qquad k = 2\ \frac{\text{kPa} \cdot \text{m}^3}{°\text{K}} \qquad \text{and consequently} \qquad P = 100\ \text{kPa}$$

If the volume and temperature are increasing at

$$\frac{dV}{dt} = 0.2\ \frac{\text{m}^3}{\text{sec}} \qquad \text{and} \qquad \frac{dT}{dt} = 6\ \frac{°\text{K}}{\text{sec}}$$

is the pressure increasing or decreasing and at what rate?

a. decreasing at $0.8\ \frac{\text{kPa}}{\text{sec}}$

b. decreasing at $1.2\ \frac{\text{kPa}}{\text{sec}}$

c. decreasing at $1.6\ \frac{\text{kPa}}{\text{sec}}$

d. decreasing at $8\ \frac{\text{kPa}}{\text{sec}}$

e. decreasing at $8.2\ \frac{\text{kPa}}{\text{sec}}$

f. increasing at $0.8\ \frac{\text{kPa}}{\text{sec}}$

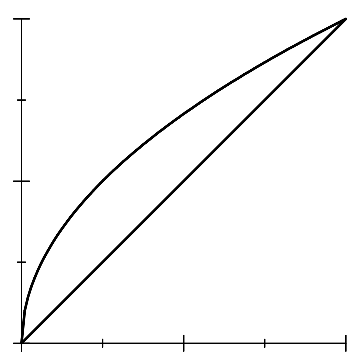
g. increasing at $1.2\ \frac{\text{kPa}}{\text{sec}}$

h. increasing at $1.6\ \frac{\text{kPa}}{\text{sec}}$

i. increasing at $8\ \frac{\text{kPa}}{\text{sec}}$

j. The pressure is constant.

7. Find the volume under the surface $z = 2x^2y$ above the region bounded by $y = x$ and $y = 2\sqrt{x}$. The base is shown at the right.



a. $\frac{160}{7}$

b. $\frac{160}{3}$

c. $\frac{64}{5}$

d. $\frac{128}{5}$

e. $\frac{48}{5}$

f. $\frac{320}{7}$

g. $\frac{320}{3}$

h. $\frac{64}{7}$

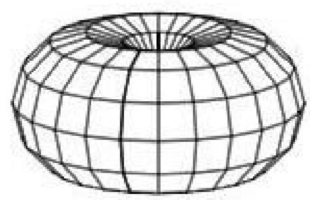
i. $\frac{256}{5}$

j. $\frac{48}{7}$

8. Find the average temperature $\bar{T} = \dfrac{\iint T\,dA}{\iint dA}$ of a circular frying pan which is 4 inches in radius if it is hottest in the center and the temperature decreases toward the edge according to $T = 133° - 3r$.

a. $58.5°$ f. $133°$
b. $62.5°$ g. $136°$
c. $125°$ h. $141°$
d. $127°$ i. $1000\pi°$
e. $132.5°$ j. $2000\pi°$

9. Which of the following integrals will give the volume of the donut given in spherical coordinates by $\rho = \sin\varphi$.



a. $\int_0^{\pi}\int_0^{\pi}\int_0^{\sin\varphi} \rho^2\cos\varphi\,d\rho\,d\varphi\,d\theta$ f. $\int_0^{\pi}\int_0^{2\pi}\int_0^{\sin\varphi} \rho^2\cos\varphi\,d\rho\,d\varphi\,d\theta$

b. $\int_0^{\pi}\int_0^{2\pi}\int_0^{1} \sin\varphi\,d\rho\,d\varphi\,d\theta$ g. $\int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\varphi} \rho^2\sin^2\varphi\,d\rho\,d\varphi\,d\theta$

c. $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sin\varphi\,\rho^2\cos\varphi\,d\rho\,d\varphi\,d\theta$ h. $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2\sin^2\varphi\,d\rho\,d\varphi\,d\theta$

d. $\int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$ i. $\int_0^{\pi}\int_0^{\pi}\int_0^{1} \rho^2\cos\varphi\,d\rho\,d\varphi\,d\theta$

e. $\int_0^{\pi}\int_0^{2\pi}\int_0^{\sin\varphi} 1\,d\rho\,d\varphi\,d\theta$ j. $\int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\varphi} 1\,d\rho\,d\varphi\,d\theta$

10. Compute $\int_{(e^{-1},1,e^{1})}^{(2e^{-2},4,8e^{2})} \vec{F} \bullet d\vec{s}$ where $\vec{F} = (yz, xz, xy)$ along the curve $\vec{r}(t) = (te^{-t}, t^2, t^3 e^t)$.
HINT: Find a scalar potential.

a. 192
b. 189
c. 126
d. 64
e. 63
f. $2e^{-3} - 4 + 8e^{3}$
g. $2e^{-3} + 4 + 8e^{3}$
h. $2e^{-2} - 6e^{-1} + 12e - 8e^{2}$
i. $32e^{2} + 8e^{-2} + e + e^{-1} + 15$
j. $32e^{2} + 8e^{-2} - e - e^{-1} + 15$

11. Compute $\oint \left(xy^2 + e^{\sqrt{x}}\right) dx + (3x^2 y + \cos(y^3))\, dy$ counterclockwise around the boundary of the region between the parabolas $y = x^2$ and $x = y^2$.
HINT: Use a theorem.

a. $\frac{-3}{2}$
b. $\frac{-2}{3}$
c. $\frac{-1}{3}$
d. $\frac{-1}{2}$
e. 0
f. $\frac{3}{2}$
g. $\frac{2}{3}$
h. $\frac{1}{3}$
i. $\frac{1}{2}$
j. 1

12. Compute $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the paraboloid $z = 9 - x^2 - y^2$ for $z \geq 0$, oriented up, for the vector field $\vec{F} = (z + y, z - x, 2z)$.

HINT: Use Stokes' Theorem. Parametrize the boundary.

a. -9π
b. -18π
c. -36π
d. -72π
e. 0
f. 9π
g. 18π
h. 36π
i. 72π
j. π

13. The surface of an apple A may be given in spherical coordinates by $\rho = 1 - \cos\varphi$ and may be parametrized by $R(\phi,\theta) = ((1-\cos\varphi)\sin\varphi\cos\theta, (1-\cos\varphi)\sin\varphi\sin\theta, (1-\cos\varphi)\cos\varphi)$.

Compute $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the apple with outward normal for $\vec{F} = (xyz^2, yzx^2, zxy^2)$.

HINT: Use Stokes' Theorem or Gauss' Theorem.

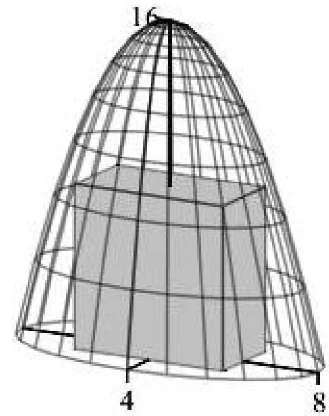
a. 0
b. -4π
c. -12π
d. $-\frac{32}{3}\pi$
e. $-\frac{64}{3}\pi$
f. π
g. 4π
h. 12π
i. $\frac{32}{3}\pi$
j. $\frac{64}{3}\pi$

Work Out: (Points indicated. Part credit possible. Show all work.)

14. (15 points) A rectangular solid sits on the xy-plane with its upper 4 vertices on the elliptic paraboloid

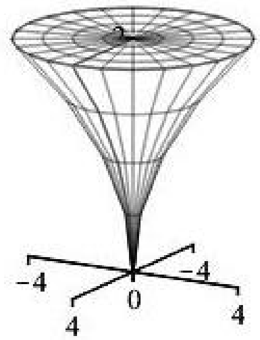
$$z = 16 - x^2 - \frac{y^2}{4}.$$

Find the dimensions and volume of the largest such box.
HINT: The length, width and height are not just x, y and z.



15. (20 points) Verify Gauss' Theorem $\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_{\partial V} \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = \langle 2xz^2, 2yz^2, -z^3 \rangle$ and the solid curved cone $0 \le r \le z^2$ with $z \le 2$.

Be careful with orientations. Use the following steps:



First the Left Hand Side:

a. Compute the divergence in rectangular coordinates:

$\vec{\nabla} \cdot \vec{F} =$

b. Express the divergence and the volume element in the appropriate coordinate system:

$\vec{\nabla} \cdot \vec{F} =$ $dV =$

c. Compute the left hand side:

$\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV =$

Second the Right Hand Side:

The boundary surface consists of a curved cone surface C and a disk D with appropriate orientations.

The disk D is at $z = 2$ with radius $r = z^2 = 4$. It may be parametrized as: $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 2)$

d. Compute the tangent vectors

$\vec{e}_r =$

$\vec{e}_\theta =$

e. Compute the normal vector:

$\vec{N} =$

f. Evaluate $\vec{F} = \langle 2xz^2, 2yz^2, -z^3 \rangle$ on the disk:

$\vec{F}\big|_{\vec{R}(r,\theta)} =$

g. Compute the dot product:

$\vec{F} \cdot \vec{N} =$

h. Compute the flux through D:

$\iint_D \vec{F} \cdot d\vec{S} =$

The curved cone surface C may be parametrized as: $\vec{R}(z,\theta) = (z^2\cos\theta, z^2\sin\theta, z)$

i. Compute the tangent vectors:

$\vec{e}_z =$

$\vec{e}_\theta =$

j. Compute the normal vector:

$\vec{N} =$

k. Evaluate $\vec{F} = \langle 2xz^2, 2yz^2, -z^3 \rangle$ on the cone:

$\vec{F}\big|_{\vec{R}(z,\theta)} =$

l. Compute the dot product:

$\vec{F} \cdot \vec{N} =$

m. Compute the flux through C:

$\iint_C \vec{F} \cdot d\vec{S} =$

n. Compute the **TOTAL** right hand side:

$\iint_{\partial V} \vec{F} \cdot d\vec{S}$.

16. (10 points) Find the mass and x-component of the center of mass of the twisted cubic **curve** $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ for $0 \leq t \leq 1$ if the density is $\rho = 3xz + 3y^2$.