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1-13	/65	15	/20
14	/15	16	/10
		Total	/110

MATH 221 Final Exam 504 Fall 2021

Sections 504/505 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the equation of the plane tangent to the graph of $z = f(x,y) = x^3y + xy^2$ at $(2,1)$. Its z-intercept is

a. -10
b. -28 Correct Choice
c. -35
d. -38
e. -48
f. 10
g. 28
h. 35
i. 38
j. 48

Solution: $f(x,y) = x^3y + xy^2$ $\quad f_x(x,y) = 3x^2y + y^2$ $\quad f_y(x,y) = x^3 + 2xy$

$f(2,1) = 10$ $\quad f_x(2,1) = 13$ $\quad f_y(2,1) = 12$

$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 10 + 13(x-2) + 12(y-1) = 13x + 12y - 28 \qquad c = -28$

2. Find an equation of the plane tangent to the surface $x^2y^2 + x^2z^2 + y^2z^2 = 49$ at the point $(1,2,3)$.

a. $13x + 10y + 5z = 38$
b. $39x + 40y + 30z = 209$
c. $13x + 20y + 15z = 98$ Correct Choice
d. $26x + 40y + 30z = 96$
e. $39x + 20y + 5z = 94$
f. $13x - 10y + 5z = 8$
g. $39x - 40y + 30z = 49$
h. $13x - 20y + 15z = 18$
i. $26x - 40y + 30z = 36$
j. $39x - 20y + 5z = 54$

Solution: $f = x^2y^2 + x^2z^2 + y^2z^2 \qquad P = (1,2,3)$

$\vec{\nabla}f = (2xy^2 + 2xz^2, 2yx^2 + 2yz^2, 2zx^2 + 2zy^2) \qquad \vec{N} = \vec{\nabla}f(P) = (26,40,30)$

$\vec{N} \cdot X = \vec{N} \cdot P \qquad 26x + 40y + 30z = 26 \cdot 1 + 40 \cdot 2 + 30 \cdot 3 = 196 \qquad 13x + 20y + 15z = 98$

3. You are standing at the airport facing North. You look up and see an airplane circling clockwise above the airport. At the moment when the plane is heading due East, in what direction does the plane's binormal point?

a. North
b. South
c. East
d. West
e. Up   Correct Choice
f. Down

$\vec{T}$ points East. Since the plane is circling clockwise, $\vec{N}$ points North. So $\vec{B} = \vec{T} \times \vec{N}$ points Up.

4. Find a parametric equation of the line tangent to the curve $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, \theta)$ at the point $(-2, 0, \pi)$.

HINT: What are the point and direction vector?

a. $X(t) = (0, -2-2t, \pi+t)$
b. $X(t) = (-2t, -2, 1+\pi t)$
c. $X(t) = (-2t, 2, 1+\pi t)$
d. $X(t) = (-2-2t, 0, \pi+t)$
e. $X(t) = (-2-2t, 0, 1+\pi t)$
f. $X(t) = (-2, -2t, \pi+t)$ Correct Choice
g. $X(t) = (-2, 2t, \pi+t)$
h. $X(t) = (2, -2t, 1+\pi t)$
i. $X(t) = (2, 2t, \pi+t)$
j. $X(t) = (2, 2t, \pi-t)$

Solution: $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, \theta) = (-2, 0, \pi)$ at $\theta = \pi$.

$\vec{v}(\theta) = (-2\sin\theta, 2\cos\theta, 1) \qquad \vec{v}(\pi) = (0, -2, 1)$

$X(t) = (-2, 0, \pi) + t(0, -2, 1) = (-2, -2t, \pi+t)$

5. Find the arc length of the curve $\vec{r}(t) = (\ln t, 2t, t^2)$ between $(0, 2, 1)$ and $(1, 2e, e^2)$.

Hint: Look for a perfect square.

a. e^2 Correct Choice
b. 1
c. $1+e$
d. $1+e^2$
e. e^2-1
f. 2
g. $2+e$
h. $e-2$
i. e^2-2
j. $2+e^2$

Solution: $\vec{v} = \left(\frac{1}{t}, 2, 2t\right) \qquad |\vec{v}| = \sqrt{\frac{1}{t^2} + 4 + 4t^2} = \frac{1}{t} + 2t$

$\vec{r}(t) = (0, 2, 1)$ at $t = 1 \qquad \vec{r}(t) = (1, 2e, e^2)$ at $t = e$

$L = \int_1^e |\vec{v}|\,dt = \int_1^e \left(\frac{1}{t} + 2t\right) dt = [\ln t + t^2]_1^e = (\ln e + e^2) - (\ln 1 + 1) = e^2$

6. For an ideal gas, the pressure, P, is a function of the temperature, T, and volume, V, given by $P = \frac{kT}{V}$ where k is a constant. For a certain sample of gas the current values are

$$T = 250°\text{K} \qquad V = 5\ \text{m}^3 \qquad k = 2\ \frac{\text{kPa} \cdot \text{m}^3}{°\text{K}} \qquad \text{and consequently} \qquad P = 100\ \text{kPa}$$

If the volume and temperature are increasing at

$$\frac{dV}{dt} = 0.2\ \frac{\text{m}^3}{\text{sec}} \qquad \text{and} \qquad \frac{dT}{dt} = 6\ \frac{°\text{K}}{\text{sec}}$$

is the pressure increasing or decreasing and at what rate?

a. decreasing at $0.8\ \frac{\text{kPa}}{\text{sec}}$

b. decreasing at $1.2\ \frac{\text{kPa}}{\text{sec}}$

c. decreasing at $1.6\ \frac{\text{kPa}}{\text{sec}}$ Correct Choice

d. decreasing at $8\ \frac{\text{kPa}}{\text{sec}}$

e. decreasing at $8.2\ \frac{\text{kPa}}{\text{sec}}$

f. increasing at $0.8\ \frac{\text{kPa}}{\text{sec}}$

g. increasing at $1.2\ \frac{\text{kPa}}{\text{sec}}$

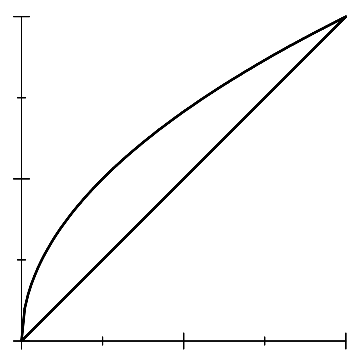
h. increasing at $1.6\ \frac{\text{kPa}}{\text{sec}}$

i. increasing at $8\ \frac{\text{kPa}}{\text{sec}}$

j. The pressure is constant.

Solution: $$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{k}{V}\frac{dT}{dt} - \frac{kT}{V^2}\frac{dV}{dt} = \frac{2 \cdot 6}{5} - \frac{2 \cdot 250 \cdot 0.2}{25} = -1.6\ \frac{\text{kPa}}{\text{sec}}$$

7. Find the volume under the surface $z = 2x^2y$ above the region bounded by $y = x$ and $y = 2\sqrt{x}$. The base is shown at the right.



a. $\frac{160}{7}$

b. $\frac{160}{3}$

c. $\frac{64}{5}$

d. $\frac{128}{5}$

e. $\frac{48}{5}$

f. $\frac{320}{7}$

g. $\frac{320}{3}$

h. $\frac{64}{7}$

i. $\frac{256}{5}$ Correct Choice

j. $\frac{48}{7}$

Solution: The curves intersect when $x = 2$ or $x^2 = 4x$ or $x = 0, 4$

$$V = \int_0^4 \int_x^{2\sqrt{x}} 2x^2 y\, dy\, dx = \int_0^4 [x^2y^2]_{y=x}^{2\sqrt{x}}\, dx = \int_0^4 (4x^3 - x^4)\, dx = \left[x^4 - \frac{x^5}{5}\right]_{x=0}^4 = \frac{4^4}{5} = \frac{256}{5}$$

8. Find the average temperature $\bar{T} = \dfrac{\iint T\,dA}{\iint dA}$ of a circular frying pan which is 4 inches in radius if it is hottest in the center and the temperature decreases toward the edge according to $T = 133° - 3r$.

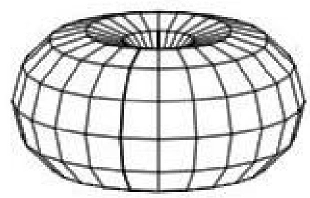
a. $58.5°$	f. $133°$
b. $62.5°$	g. $136°$
c. $125°$ Correct Choice	h. $141°$
d. $127°$	i. $1000\pi°$
e. $132.5°$	j. $2000\pi°$

Solution: $A = \iint dA = \int_0^{2\pi}\int_0^4 r\,dr\,d\theta = 2\pi\left[\frac{r^2}{2}\right]_0^4 = 16\pi$ OR $A = \pi R^2 = \pi 4^2 = 16\pi$

$$\iint T\,dA = \int_0^{2\pi}\int_0^4 (133 - 3r)r\,dr\,d\theta = 2\pi\left[133\frac{r^2}{2} - r^3\right]_0^4 = 2\pi(133 \cdot 8 - 64) = 2\pi(1064 - 64) = 2000\pi$$

$$\bar{T} = \frac{1}{A}\iint T\,dA = \frac{2000\pi}{16\pi} = 125°$$

9. Which of the following integrals will give the volume of the donut given in spherical coordinates by $\rho = \sin\varphi$.



a. $\int_0^{\pi}\int_0^{\pi}\int_0^{\sin\varphi} \rho^2\cos\varphi\,d\rho\,d\varphi\,d\theta$	f. $\int_0^{\pi}\int_0^{2\pi}\int_0^{\sin\varphi} \rho^2\cos\varphi\,d\rho\,d\varphi\,d\theta$
b. $\int_0^{\pi}\int_0^{2\pi}\int_0^{1} \sin\varphi\,d\rho\,d\varphi\,d\theta$	g. $\int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\varphi} \rho^2\sin^2\varphi\,d\rho\,d\varphi\,d\theta$
c. $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \sin\varphi\,\rho^2\cos\varphi\,d\rho\,d\varphi\,d\theta$	h. $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2\sin^2\varphi\,d\rho\,d\varphi\,d\theta$
d. $\int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$ Correct Choice	i. $\int_0^{\pi}\int_0^{\pi}\int_0^{1} \rho^2\cos\varphi\,d\rho\,d\varphi\,d\theta$
e. $\int_0^{\pi}\int_0^{2\pi}\int_0^{\sin\varphi} 1\,d\rho\,d\varphi\,d\theta$	j. $\int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\varphi} 1\,d\rho\,d\varphi\,d\theta$

Solution: $V = \iiint 1\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

10. Compute $\int_{(e^{-1},1,e^{1})}^{(2e^{-2},4,8e^{2})} \vec{F}\cdot d\vec{s}$ where $\vec{F} = (yz,xz,xy)$ along the curve $\vec{r}(t) = (te^{-t},t^2,t^3e^t)$.
HINT: Find a scalar potential.

a. 192
b. 189
c. 126
d. 64
e. 63 Correct Choice
f. $2e^{-3}-4+8e^{3}$
g. $2e^{-3}+4+8e^{3}$
h. $2e^{-2}-6e^{-1}+12e-8e^{2}$
i. $32e^{2}+8e^{-2}+e+e^{-1}+15$
j. $32e^{2}+8e^{-2}-e-e^{-1}+15$

Solution: $\vec{F} = (yz,xz,xy) = \vec{\nabla}f$ for $f = xyz$. So, by the FTCC:

$$\int_{(e^{-1},1,e^{1})}^{(2e^{-2},4,8e^{2})} \vec{F}\cdot d\vec{s} = \int_{(e^{-1},1,e^{1})}^{(2e^{-2},4,8e^{2})} \vec{\nabla}f\cdot d\vec{s} = f(2e^{-2},4,8e^{2}) - f(e^{-1},1,e^{1}) = 64-1 = 63$$

11. Compute $\oint\left(xy^2+e^{\sqrt{x}}\right)dx + (3x^2y+\cos(y^3))\,dy$ counterclockwise around the boundary of the region between the parabolas $y = x^2$ and $x = y^2$.
HINT: Use a theorem.

a. $\frac{-3}{2}$
b. $\frac{-2}{3}$
c. $\frac{-1}{3}$
d. $\frac{-1}{2}$
e. 0
f. $\frac{3}{2}$
g. $\frac{2}{3}$
h. $\frac{1}{3}$ Correct Choice
i. $\frac{1}{2}$
j. 1

Solution: By Green's Theorem,

$$\oint\left(xy^2+e^{\sqrt{x}}\right)dx + (3x^2y+\cos(y^3))\,dy = \iint\left(\frac{\partial}{\partial x}(3x^2y+\cos(y^3)) - \frac{\partial}{\partial y}\left(xy^2+e^{\sqrt{x}}\right)\right)dx\,dy$$

$$= \iint(6xy-2xy)\,dx\,dy = \int_0^1\int_{x^2}^{\sqrt{x}} 4xy\,dy\,dx = \int_0^1\left[2xy^2\right]_{y=x^2}^{\sqrt{x}}dx = \int_0^1(2x^2-2x^5)\,dx$$

$$= \left[\frac{2x^3}{3}-\frac{x^6}{3}\right]_{x=0}^{1} = \frac{2}{3}-\frac{1}{3} = \frac{1}{3}$$

12. Compute $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the paraboloid $z = 9 - x^2 - y^2$ for $z \geq 0$, oriented up, for the vector field $\vec{F} = (z + y, z - x, 2z)$.

HINT: Use Stokes' Theorem. Parametrize the boundary.

a. -9π

b. -18π Correct Choice

c. -36π

d. -72π

e. 0

f. 9π

g. 18π

h. 36π

i. 72π

j. π

Solution: The boundary is the circle $x^2 + y^2 = 9$ for $z = 0$.

It may be parametrized as $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 0)$.

$\vec{v} = (-3\sin\theta, 3\cos\theta, 0)$ Oriented counterclockwise as seen from above.

$\vec{F}(\vec{r}(\theta)) = (3\sin\theta, -3\cos\theta, 0)$

$$\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial P} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -9\, d\theta = -18\pi$$

13. The surface of an apple A may be given in spherical coordinates by $\rho = 1 - \cos\varphi$ and may be parametrized by $R(\phi, \theta) = ((1 - \cos\varphi)\sin\varphi\cos\theta, (1 - \cos\varphi)\sin\varphi\sin\theta, (1 - \cos\varphi)\cos\varphi)$.

Compute $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the apple with outward normal for $\vec{F} = (xyz^2, yzx^2, zxy^2)$.

HINT: Use Stokes' Theorem or Gauss' Theorem.

a. 0 Correct Choice

b. -4π

c. -12π

d. $-\frac{32}{3}\pi$

e. $-\frac{64}{3}\pi$

f. π

g. 4π

h. 12π

i. $\frac{32}{3}\pi$

j. $\frac{64}{3}\pi$

Solution: By Stokes' Theorem, $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint \vec{F} \cdot d\vec{s} = 0$ because there is no boundary curve.

OR: By Gauss' Theorem, $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iiint \vec{\nabla} \cdot \vec{\nabla} \times \vec{F}\, dV = 0$ because $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$.

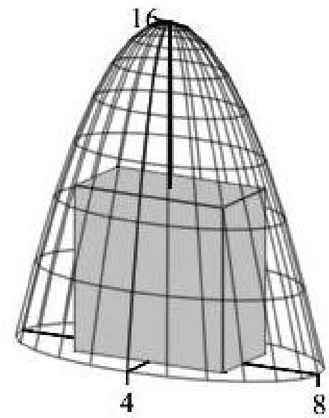
Work Out: (Points indicated. Part credit possible. Show all work.)

14. (15 points) A rectangular solid sits on the xy-plane with its upper 4 vertices on the elliptic paraboloid

$$z = 16 - x^2 - \frac{y^2}{4}.$$

Find the dimensions and volume of the largest such box.

HINT: The length, width and height are not just x, y and z.



Solution: The length, width and height are $L = 2x$, $W = 2y$ and $H = z$.

So the volume is $V = LWH = 4xyz$. The constraint is $g = z + x^2 + \frac{y^2}{4} = 16$.

The gradients are $\vec{\nabla} V = \langle 4yz, 4xz, 4xy \rangle$ and $\vec{\nabla} g = \left\langle 2x, \frac{y}{2}, 1 \right\rangle$.

The Lagrange equations are $4yz = \lambda 2x$ $\quad 4xz = \lambda \frac{y}{2}$ $\quad 4xy = \lambda$

Multiply the first equation by x, the second by y and the third by z and equate:

$$4xyz = \lambda 2x^2 = \lambda \frac{y^2}{2} = \lambda z$$

So $x^2 = \frac{z}{2}$ and $y^2 = 2z$. We plug these into the constraint:

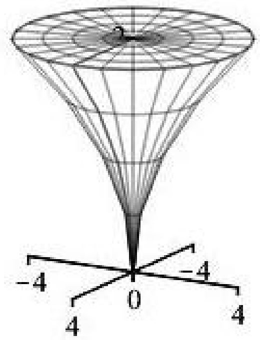
$$z + x^2 + \frac{y^2}{4} = z + \frac{z}{2} + \frac{z}{2} = 2z = 16 \quad \Rightarrow \quad z = 8$$

Then $x^2 = 4$ $\quad y^2 = 16$ or $x = 2$ $\quad y = 4$

The dimensions are $L = 4$, $W = 8$ and $H = 8$ and the volume is $V = 4 \cdot 8 \cdot 8 = 256$.

15. (20 points) Verify Gauss' Theorem $\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \iint\limits_{\partial V} \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = \langle 2xz^2, 2yz^2, -z^3 \rangle$ and the solid curved cone $0 \le r \le z^2$ with $z \le 2$.

Be careful with orientations. Use the following steps:



First the Left Hand Side:

a. Compute the divergence in rectangular coordinates:

Solution: $\vec{\nabla} \cdot \vec{F} = 2z^2 + 2z^2 - 3z^2 = z^2$

b. Express the divergence and the volume element in the appropriate coordinate system:

Solution: $\vec{\nabla} \cdot \vec{F} = z^2$ $\qquad dV = r\,dr\,d\theta\,dz$

c. Compute the left hand side:

Solution: $\iiint\limits_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi}\int_0^2\int_0^{z^2} z^2 r\,dr\,dz\,d\theta = 2\pi \int_0^2 z^2 \left[\frac{r^2}{2}\right]_0^{z^2} dz = \pi \int_0^2 z^6\, dz = \pi\left[\frac{z^7}{7}\right]_0^2 = \frac{128}{7}\pi$

Second the Right Hand Side:

The boundary surface consists of a curved cone surface C and a disk D with appropriate orientations.

The disk D is at $z = 2$ with radius $r = z^2 = 4$. It may be parametrized as: $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 2)$

d. Compute the tangent vectors:

Solution: $\vec{e}_r = \langle \cos\theta, \quad \sin\theta, \quad 0 \rangle$

$\vec{e}_\theta = \langle -r\sin\theta, \quad r\cos\theta, \quad 0 \rangle$

e. Compute the normal vector:

Solution: $\vec{N} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = \langle 0, 0, r \rangle$

This points up as required.

f. Evaluate $\vec{F} = \langle 2xz^2, 2yz^2, -z^3 \rangle$ on the disk:

Solution: $\vec{F}\Big|_{\vec{R}(r,\theta)} = \langle 8r\cos\theta, 8r\sin\theta, -8 \rangle$

g. Compute the dot product:

Solution: $\vec{F} \cdot \vec{N} = -8r$

h. Compute the flux through D:

Solution: $\iint_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^4 \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^4 -8r\, dr\, d\theta = -2\pi[4r^2]_0^4 = -2\pi 4^3 = -128\pi$

The curved cone surface C may be parametrized as: $\vec{R}(z,\theta) = (z^2\cos\theta, z^2\sin\theta, z)$

i. Compute the tangent vectors:

Solution: $\vec{e}_z = \left\langle 2z\cos\theta, \quad 2z\sin\theta, \quad 1\right\rangle$

$\vec{e}_\theta = \langle -z^2\sin\theta, \quad z^2\cos\theta, \quad 0\rangle$

j. Compute the normal vector:

Solution: $\vec{N} = \hat{\imath}(-z^2\cos\theta) - \hat{\jmath}(--z^2\sin\theta) + \hat{k}(2z^3\sin^2\theta - -2z^3\cos^2\theta) = \langle -z^2\cos\theta, -z^2\sin\theta, 2z^3\rangle$

This is up and in. We need down and out.

Reverse: $\vec{N} = \langle z^2\cos\theta, z^2\sin\theta, -2z^3\rangle$

k. Evaluate $\vec{F} = \langle 2xz^2, 2yz^2, -z^3\rangle$ on the cone:

Solution: $\vec{F}\Big|_{\vec{R}(z,\theta)} = \langle 2z^4\cos\theta, 2z^4\sin\theta, -z^3\rangle$

l. Compute the dot product:

Solution: $\vec{F}\cdot\vec{N} = 2z^6\cos^2\theta + 2z^6\sin^2\theta + 2z^6 = 4z^6$

m. Compute the flux through C:

Solution: $\iint_C \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^2 \vec{F}\cdot\vec{N}\,dz\,d\theta = \int_0^{2\pi}\int_0^2 4z^6\,dz\,d\theta = 8\pi\left[\frac{z^7}{7}\right]_0^2 = 8\pi\frac{128}{7}$

n. Compute the **TOTAL** right hand side:

Solution: $\iint_{\partial V} \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot d\vec{S} + \iint_C \vec{F}\cdot d\vec{S} = -128\pi + 8\pi\frac{128}{7} = 128\pi\left(-1 + \frac{8}{7}\right) = \frac{128\pi}{7}$

which agrees with (c).

16. (10 points) Find the mass and x-component of the center of mass of the twisted cubic **curve** $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ for $0 \le t \le 1$ if the density is $\rho = 3xz + 3y^2$.

Solution: $\vec{v} = (1, 2t, 2t^2)$ $\quad |\vec{v}| = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2$ $\quad \rho = 3t\frac{2}{3}t^3 + 3(t^2)^2 = 5t^4$

$$M = \int \rho\,ds = \int_0^1 \rho|\vec{v}|\,dt = \int_0^1 5t^4(1+2t^2)\,dt = 5\int_0^1 (t^4 + 2t^6)\,dt = 5\left[\frac{t^5}{5} + 2\frac{t^7}{7}\right]_0^1 = 5\left(\frac{1}{5} + \frac{2}{7}\right) = \frac{17}{7}$$

$$M_x = \int x\rho\,ds = \int_0^1 x\rho|\vec{v}|\,dt = \int_0^1 t5t^4(1+2t^2)\,dt = 5\int_0^1 (t^5 + 2t^7)\,dt$$

$$= 5\left[\frac{t^6}{6} + 2\frac{t^8}{8}\right]_0^1 = 5\left(\frac{1}{6} + \frac{1}{4}\right) = \frac{25}{12}$$

$$\bar{x} = \frac{M_x}{M} = \frac{25}{12}\frac{7}{17} = \frac{175}{204}$$